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ON THE THEOREMS OF Y. MIBU AND G. DEBS ON SEPARATE CONTINUITY

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ABSTRACT. Using a game-theoretic characterization of Baire spaces, conditions upon the domain and the range are given to ensure a "fat" set $C(f)$ of points of continuity in the sets of type $X \times \{y\}$, $y \in Y$ for certain almost separately continuous functions $f : X \times Y \to Z$. These results (especially Theorem B) generalize Mibu's First Theorem, previous theorems of the author, answers one of his problems as well as they are closely related to some other results of Debs [1] and Mibu [2].

KEY WORDS AND PHRASES: Separate and joint continuity, quasi-continuity, Moore spaces.
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1. INTRODUCTION

Since the appearance of the celebrated result of I. Namioka, many articles have been written on the topic of separate and joint continuity, see Piotrowski [3], for a survey.

Aside from an intensively studied Uniformization Problem – Namioka-type theorems, see Piotrowski [3], questions pertaining to Existence Problem (see below) as well as its generalizations, have been asked.

Let X and Y be "nice" (e.g. Polish) topological spaces, let M be metric and let $f : X \times Y \to M$ be separately continuous, that is, continuous with respect to each variable while the other is fixed. Find the set $C(f)$ of points of (joint) continuity of f.

Let us recall that given spaces X, Y and Z, and let $f : X \times Y \to Z$ be a function. For every fixed $x \in X$, the function $f_x : Y \to Z$ defined by $f_x(y) = f(x, y)$, where $y \in Y$, is called an <u>x-section</u> of f. A y-section f_y of f is defined similarly.

One way to ensure the existence of "many" points of continuity in $X \times Y$ can be derived from the following. <u>Baire-Lebesgue-Kuratowski-Montgomery Theorem</u> (see Piotrowski [3]). Let X and Y be metric and let $f : X \times Y \to \mathbb{R}$ have all x-sections f_x continuous and have all y-sections f_y of Baire class α. Then f is of class $a + 1$.

If $\alpha = 0$, that is, f_y is continuous, f is of class 1. Now, by a theorem of Baire, $C(f)$ is residual. So, if we assume additionally that $X \times Y$ is Baire, then $C(f)$ is a dense G_δ subset of $X \times Y$. □

But one cannot relax the assumptions pertaining to the sections too much.

EXAMPLE. Let $I = [0,1]$ and let $\mathbb{R}$ be the set of reals. Put $D_n = \{(x,y) : x = \frac{k}{2^n}, y = \frac{p}{2^n}$, where k and p are all odd numbers between 0 and $2^n\}$. Let $D = \cup_{n=1}^{\infty} D_n$. It is easy to see that Cl $D = I^2$. Now, let us define $f : I^2 \to \mathbb{R}$ by: $f(x,y) = 1$, for $(x,y) \in D$ and $f(x,y) = 0$ if $(x,y) \notin D$. All the x-sections f_x and all the y-sections f_y are of first class of Baire and $C(f) = \phi$. □

However, the following three important results hold.

MIBU'S FIRST THEOREM (Mibu [2]). Let X be first countable, Y be Baire and such that $X \times Y$ is Baire. Given a metric space M. If $f : X \times Y \to M$ is separately continuous, then $C(F)$ is a dense G_δ subset of $X \times Y$.

MIBU'S SECOND THEOREM [2]. Let X be second countable, Y be Baire and such that $X \times Y$ is Baire. Given a metric space M. If $f : X \times Y \to M$ has:

a) all x-sections f_x have their sets $D(f)$ of points of discontinuity of the first category and;

b) all y-sections f_y are continuous.

Then $C(f)$ is a dense, G_δ subset of $X \times Y$.

Following Debs [1], a function $f : X \to M$ is called _first class_ if for every $\epsilon > 0$, for every nonempty subset $A \subset X$, there is a nonempty set U, open in A, such that diam $(f(U)) \leq \epsilon$.

DEBS' THEOREM [1]. Let X be first countable Y be a special α-favorable space (thus Baire), $X \times Y$ be Baire. Given a metric space M. If $f : X \times Y \to M$ has:

a) all x-sections f_x of first class – in the sense of Debs and;

b) all y-sections f_y continuous.

Then $C(f)$ is a dense G_δ subset of $X \times Y$.

2. QUASI-CONTINUITY ON PRODUCT SPACES

A function $f : X \to Y$ is called _quasi-continuous at a point_ $x \in X$ if for each open sets $A \subset X$ and $H \subset f(X)$, where $x \in A$ and $f(x) \in H$, we have $A \cap Int\, f^{-1}(H) \neq \emptyset$. A function $f : X \to Y$ is called _quasi-continuous_, if it is quasi-continuous at each point x of X.

A function $f : X \times Y \to Z$ (X, Y, Z – arbitrary topological spaces) is said to be _quasi-continuous at $(p,q) \in X \times Y$ with respect to the variable y_, if for every neighborhood N of $f(p,q)$ and for every neighborhood $U \times V$ of (p,q), there exists a neighborhood V' of q, with $V' \subset V$, and a nonempty open $U' \subset U$, such that for all $(x,y) \in U' \times V'$ we have $f(x,y) \in N$. If f is quasi-continuous with respect to the variable y at each point of its domain, it will be called _quasi-continuous with respect to y_. The definition of a function f that is quasi-continuous with respect to x is quite similar. If f is quasi-continuous with respect to x and y, we say that f is _symmetrically quasi-continuous_.

One can easily show from the definitions that if f is symmetrically quasi-continuous, then f_x and f_y are quasi-continuous for all $x \in X$ and $y \in Y$. The converse does not hold.

LEMMA (Piotrowski [4] Theorem 4.2). Let X be a Baire space, Y be first countable and Z be regular. If f is a function on $X \times Y$ to Z such that all its x-sections f_x are continuous and all its y-sections f_y are quasi-continuous, then f is quasi-continuous with respect to y.

The converse does not hold.

As an immediate consequence we obtain (Piotrowski [4] Corollary 4.3). Let X and Y be first countable, Baire spaces and Z be a regular one. If $f : X \times Y \to Y$ is separately continuous, then f is symmetrically quasi-continuous.

If X and Y are second countable. Baire spaces and Z is a regular one, and a function $f : X \times Y \to Z$, then the following implications hold (which show the inclusion relations between proper classes of functions) – see Diagram 1. None of these implications can, in general be replaced by an equivalence, see Neubrunn [5].

		f-continuous		
	↙	↓	↘	
f-symmetrically quasi-continuous	←	f_x, f_y-continuous	→	f-quasi-continuous
	↘	↓	↗	
		f_x, f_y-quasi-continuous		

Diagram 1

The Banach-Mazur game. We will use here the classical Banach-Mazur game between players A and B both playing with perfect information (see Noll [6], Oxtoby [7]). A ***strategy*** for player A is a mapping α whose domain is the set of all decreasing sequences $(G_1, ..., G_{2n-1})$, $n \geq 1$, of nonempty open sets such that $\alpha(G_1, ..., G_{2n-1})$ is a nonempty open set contained in G_{2n-1}. Dually, a strategy for player B is a mapping β whose domain is the set of all decreasing sequences $(U_1, ..., U_{2n})$, $n \geq 0$, of nonempty open sets such that $\beta(U_1, ..., U_{2n})$ is nonempty, open and contained in U_{2n}. Here $n = 0$ stands for the empty sequence, for which $\beta(\emptyset)$ is nonempty and open, too. If α, β are strategies for A, B respectively, then the unique sequence $G_1, G_2, G_3, ...$ defined by $\beta(\emptyset) = G_1$, $\alpha(G_1) = G_2$, $\beta(G_1, G_2) = G_3$, $\alpha(G_1, G_2, G_3) = G_4, ...$ is called the game of A with α against B with β. We will say that A with α wins against B with β if $\cap \{G_n : n \in N\} \neq \emptyset$ holds for the game $G_1, G_2, ...$ of A with α against B with β. Conversely, we will say that B with β wins against A with α if A with α does not win against B with β.

We will make use of the following theorem, essentially proved by Banach and Mazur cf. Oxtoby [7], see also Noll [6] where the game-theoretic characterization of Baire spaces was applied to obtain some graph theorems.

Let E be a topological space. The following are equivalent:

(1) E is a Baire space;

(2) for every strategy β of B there exists a strategy α of A with α wins against B with β.

3. THE MAIN RESULT

Let us recall: If $A \subset X$ and $\mathcal{U}$ is a collection of subsets of X, then $st.(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. For $x \in X$, we write $st.(x, \mathcal{U})$ instead of $st.(\{x\}, \mathcal{U})$. A sequence $\{G_n\}$ of open covers of X is a *development* of X if for each $x \in X$ the set $\{st.(x, G_n) : n \in N\}$ is a base at x. A *developable* space is a space which has a development. A *Moore space* is a regular developable space.

THEOREM A. Let X be a Baire space, Y be space and let $\{P_n\}_{n=1}^{\infty}$ be a development for Z. If $f : X \times Y \to Z$ is quasi-continuous with respect to y, then $C(f)$ is a dense, G_δ subset in $X \times \{y\}$, for all $y \in Y$.

PROOF. Let $x \in X$, $y \in Y$ and let $U \times V$ be a neighborhood of (x, y). Define a strategy β for a player B in a corresponding Banach-Mazur game played over X. For this purpose we shall order (well-ordering) the sets: X, open neighborhoods of y and open nonempty subsets of X.

(1) $\beta(\emptyset)$ has to be defined. Since Z has a countable development $\mathcal{P}_n$, there is a local countable base at every point of Z; in particular take $\{G_n\}$ at $f(x, y)$. Pick G_1. Now by the quasi-continuity of f with respect to y, there is a neighborhood V^1 of y, and a nonempty open U^1 such that $f(U^1 \times V^1) \subset G_1$. Let us further assume that U^1 and V^1 are the *first sets* in their orderings of X and Y, respectively with the above property. Now, let W^1 be the first nonempty open set contained in U^1 and let x_1 be the first element of W^1. Thus, $W^1 \times V^1$ is a neighborhood of (x_1, y). So, let $\beta(\emptyset) = W^1$.

(2) $\beta(G_1, G_2)$ has to be defined, where G_1, G_2 are nonempty open and $G_2 \subset G_1$. Now, f is quasi-continuous with respect to y at (x_2, y), pick G^3, the first element of the base at $f(x_1, y)$ with $G_3 \subset G_2$. Now pick the first element $U^3 \times V^3$ such that $f(U^3 \times V^3) \subset G_3$ – such a $U^3 \times V^3$ exists, by the quasi-continuity with respect to y of f. Now, let W^3 be the first open nonempty set contained in U^3 (a priori, it can be even the same set (!)) and let $x_3 \in U^3$ be the first element of W^3. Thus, $W^3 \times V^3$ is a neighborhood of (x_3, y). So, let $\beta(G_1, G_2) = W^3$.

(3) In this way we proceed to define β by recursion, i.e., if $\beta(\emptyset) = G_1$ and $\beta(G_1, ..., G_{2k}) = G_{2k+1}$, for all $k < n$ then the former steps are available and we can define G_{2k+1} in analogy with (2).

(4) Suppose now that β has been defined. Since X is Baire, there is a strategy α for A such that A with α wins against B with β (see the definition of the game).
Let $G_1, G_2 ...$ be the game A with α against B with β.
Notice that:

$$\bigcap\{W_n : n \in N\} = \bigcap\{G_n : n \in N\}. \tag{3.1}$$

But observe that α is winning, hence this intersection is nonempty; i.e., $x^* \in \bigcap\{W_n : n \in N\}$, so $(x^*, y) \in (U \times V) \cup (X \times \{y\})$. This in turn shows the density of $C(f)$ in $X \times \{y\}$. The G_δ part follows easily from the construction. □

A space will be called *quasi-regular* if for every nonempty open set U, there is a nonempty open set V such that $\mathrm{Cl}\, V \subset U$. Obviously, every regular space is quasi-regular.

Let $\mathcal{A}$ be an open covering of a space X. Then a subset S of X is said to be *$\mathcal{A}$-small* if S is contained in a member of $\mathcal{A}$. A space X is said to be *strongly countably complete* if there is a sequence

$\{A_i : i = 1, 2, ...\}$ of open coverings of X such that a sequence $\{F_i\}$ of closed subsets of X has a nonempty intersection provided that $F_i \supset F_{i+1}$ for all i and each F_i is $\mathcal{A}_i$-small.

The class of strongly countably complete spaces includes locally countable compact spaces and complete metric spaces.

In view of the following: (Piotrowski [8], Theorem 4.6, see also Lemma 3 of Piotrowski [9]). Every quasi-regular, strongly countably complete space X is a Baire space.

Theorem A is a strong generalization of the following:

(Piotrowski [8], Theorem 4.5). Let X be a space, Y be quasi-regular, strongly countably complete and Z be metric. If $f : X \times Y \to Z$ is quasi-continuous with respect to x, then for all $x \in X$ the set of points of joint continuity of f is a dense G_δ of $\{x\} \times Y$. Further, observe that our Theorem A answers, in positive, the following:

(Piotrowski [8], Problem 4.11). Does Theorem 4.5 (of [8]) hold if Y is only assumed to be a quasi-regular Baire space?

The following Theorem B is the main result of this paper and its proof easily follows from the lemma and Theorem A.

THEOREM B. Let X be first countable, Y be Baire and Z be Moore. If $f : X \times Y \to Z$ has all its x-sections f_x quasi-continuous and all its y-sections f_y continuous, then for all $x \in X$, the set of points of continuity of f is a dense G_δ subset of $\{x\} \times Y$.

The above result strongly generalizes (see the assumptions upon Y and Z) the following known theorem:

(Piotrowski [8], Theorem 4.8, see also Theorem 5 of Piotrowski [9]). Let X be first countable, Y be strongly countably complete, quasi-regular and Z be a metric space. If $f : X \times Y \to Z$ is a function such that all its x-sections f_x are quasi-continuous and all its y-sections f_y are continuous, then for all $x \in X$, the set of points of joint continuity of f is a dense G_δ subset of $\{x\} \times Y$.

Our Theorem B generalizes in many ways Mibu's First Theorem – see Introduction.

It is also closely *related* to Mibu's Second Theorem and Debs' Theorem – ibidem. Observe though, that quasi-continuity of a function does not imply – nor is implied, by the condition of being of first class – in the sense of Debs.

Really, let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = 0$, if $x \neq \frac{1}{2}$. Then such a function f is of first class, in the sense of Debs and, clearly, it is not quasi-continuous.

There are quasi-continuous functions $f : \mathbb{R} \to \mathbb{R}$ which are of arbitrary class of Baire or not Lebesgue measurable – see Neubrunn [5] for more details.

REMARK 1. The studies of the continuity points of functions whose ranges are not necessarily metric have been done already in the 1960's, see Klee and Schwarz [10] or later in the 1980's, see Dubins [11], we omit here an extensive literature of this approach, when the range is a uniform space.

REMARK 2. Recently, the author has obtained some results of this paper using though entirely different techniques, see Piotrowski [12].

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