On Volterra Spaces II

DAVID GAULD,[a] SINA GREENWOOD,[a] AND ZBIGNIEW PIOTROWSKI[b]

[a]Department of Mathematics
University of Auckland
Private Bag 92019
Auckland, New Zealand
and
[b]Department of Mathematics
Youngstown State University
Youngstown, Ohio 44555

ABSTRACT: We say that a topological space X is Volterra if for each pair $f, g : X \to \mathbb{R}$ for which the sets of points at which f, respectively g, are continuous are dense, there is a common point of continuity; and X is strongly Volterra if in the same circumstances the set of common points of continuity is dense in X. For both of these concepts equivalent conditions are given and the situation involving more than two functions is explored.

For any function $f: X \to Y$ we denote by $C(f)$ the set of points of X at which f is continuous. Motivated by the proof in [3] and observations in [1] we make the following definitions.

DEFINITION 1: Let X be a topological space. Say that X is *Volterra* if $C(f) \cap C(g) \neq \emptyset$ whenever $f, g : X \to \mathbb{R}$ are two functions for which $C(f)$ and $C(g)$ are dense in X.

DEFINITION 2: Let X be a topological space. Say that X is *strongly Volterra* if $C(f) \cap C(g)$ is dense in X whenever $f, g : X \to \mathbb{R}$ are two functions for which $C(f)$ and $C(g)$ are dense in X.

It should be noted that these definitions conflict with those given in [2] where what we call strongly Volterra is called Volterra. However, in Theorem 2 below we show that the two concepts, Volterra and strongly Volterra, defined in [2] are in fact equivalent. The definitions given here conform more closely to Volterra's original proof.

LEMMA: (Cf. Proposition 18 of [2].) Let X be a topological space and A a dense G_δ subset of X. Then there is a function $f: X \to \mathbb{R}$ for which $C(f) = A$.

Proof: As A is G_δ we may write $A = \bigcap_{i=1}^{\infty} A_i$ where each A_i is open,

Mathematics Subject Classification: Primary 54E99; Secondary 26A15, 54C05, 54C30, 54E52.
Key words and phrases: Volterra spaces, strongly Volterra spaces, Baire spaces, developable spaces, sets of continuity points.

$A_{n+1} \subset A_n$ for each n and $A_1 = X$. Define $f: X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1/n & \text{if } x \in A_n - A_{n+1}. \end{cases}$$

We have $C(f) \subset A$ for if $x \notin A$ then $x \in A_n - A_{n+1}$ for some n. Then $(0, \infty)$ is a neighborhood of $f(x)$ but $f^{-1}(0, \infty)$ cannot be a neighborhood of x as it contains no point of the dense set A. Thus $x \notin C(f)$. Conversely, it is clear that $A \subset C(f)$. □

THEOREM 1: For any topological space X, the following three conditions are equivalent:

(a) X is Volterra;

(b) for each pair A, B of dense G_δ subsets of X the set $A \cap B$ is nonempty;

(c) for each pair Y, Z of developable spaces and each pair $f: X \to Y$ and $g: X \to Z$ of functions for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap C(g)$ is nonempty.

Proof: (a) ⇒ (b). Suppose A, B are dense G_δ subsets of X. By the lemma there are functions $f, g: X \to \mathbb{R}$ with $C(f) = A$ and $C(g) = B$. The result is immediate from the definition.

(b) ⇒ (c). Suppose Y and Z are developable spaces and $f: X \to Y$ and $g: X \to Z$ are such that $C(f)$ and $C(g)$ are dense. By Remark 6 of [2], both $C(f)$ and $C(g)$ are G_δ sets so by (b) $C(f) \cap C(g) \neq \varnothing$.

(c) ⇒ (a). This follows from the developability of $\mathbb{R}$. □

THEOREM 2: For any topological space X, the following three conditions are equivalent:

(a) X is strongly Volterra;

(b) for each pair A, B of dense G_δ subsets of X the set $A \cap B$ is dense;

(c) for each pair Y, Z of developable spaces and each pair $f: X \to Y$ and $g: X \to Z$ of functions for which $C(f)$ and $C(g)$ are dense in X the set $C(f) \cap C(g)$ is dense.

Proof: Similar to the proof of Theorem 1. □

THEOREM 3: Let X be a nonempty topological space.

(a) If X is Baire, then X is strongly Volterra.

(b) If X is strongly Volterra, then X is Volterra.

(c) If X is of second category, then X is Volterra.

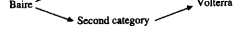
Proof:

(a) This is the content of Theorem 8 of [2].

(b) Obvious.

(c) This follows from (a) and Theorem 1, since any space of second category contains a nonempty open Baire subspace. ❑

Theorem 3 may be summarized by the following diagram where the arrows represent implications. (It is known that a Baire space is of second category.)



The following examples show that the converse implications in Theorem 3 fail, even when X is a T_1-space, and that there is no relation between strongly Volterra spaces and spaces of second category.

EXAMPLE 1: Let $X = [0, \infty)$ with the following topology:

$$\{\phi\} \cup \{[a, \infty) - F / a \in X \text{ and } F \text{ is a finite subset of } X\}.$$

X is not of second category (hence, not a Baire space) because X is the union of countably many closed, nowhere dense sets $\{[0, n)/n \in \mathbb{N}\}$. However, each dense G_δ set is of the form $[a, \infty) - C$ where $a \in X$ and C is a countable subset of X and the intersection of any pair of such sets is of the same form, hence dense. Thus X is strongly Volterra (and, therefore, also Volterra). Note that singletons are closed so X is T_1.

EXAMPLE 2: Let $X = \mathbb{R}_- \cup \mathbb{Q}_+$ topologized as a subspace of the reals, where $\mathbb{R}_- = \{x \in \mathbb{R} / x \leq 0\}$ and $\mathbb{Q}_+ = \{x \in \mathbb{Q} / x \geq 0\}$. Then X is not strongly Volterra because if we let $\mathbb{Q}_{+o}$ and $\mathbb{Q}_{+e}$ denote, respectively, the members of $\mathbb{Q}_+$ with odd and even denominators when in lowest terms then $\mathbb{R}_- \cup \mathbb{Q}_{+o}$ and $\mathbb{R}_- \cup \mathbb{Q}_{+e}$ are dense G_δ subsets of X whose intersection is not dense. On the other hand X is Volterra because if $f, g: X \to \mathbb{R}$ are such that $C(f)$ and $C(g)$ are dense in X then $C(f|\mathbb{R}_-)$ and $C(g|\mathbb{R}_-)$ are dense in the strongly Volterra space $\mathbb{R}_-$, hence have nonempty intersection. X is also of second category.

Note that in Example 2, X is metrizable. Is there a metrizable example of a strongly Volterra, non-Baire space, or better still, a non-Baire subspace of the reals? As a possible contender, illustrating a possible way of constructing such a space, we offer the following. Let C be the Cantor set and C_n the Cantor set translated by n ($n \in \mathbb{Z}$). Let $U_1 = \mathbb{R} - \bigcup_{n=-\infty}^{\infty} C_n$. Define U_n ($n \in \mathbb{N}$) inductively by $U_n = \{x \in \mathbb{R} / 3x \in U_{n-1}\}$. Then each U_n is open and dense in $\mathbb{R}$ so $\bigcap_{n=1}^{\infty} U_n$ is a dense G_δ. Let $X = \mathbb{R} - \bigcap_{n=1}^{\infty} U_n$. Then X is not Baire because if $X_n = X \cap U_n$, then X_n is open in X. X_n is dense in $\mathbb{R}$, being the intersection of two dense sets one of which is open. Thus X_n is a dense open subset of X. However, $\bigcap_{n=1}^{\infty} X_n = \emptyset$. Thus, $\{X_n / n \in \mathbb{N}\}$ is a countable collection of dense open subsets of X having empty, hence nondense, intersection. Therefore, X is not Baire.

As a corollary of Theorem 2, we have the following result.

THEOREM 4: Let X be a strongly Volterra space, Y_i ($i \in \mathbb{N}$) developable spaces and $f_i: X \to Y_i$ ($i \in \mathbb{N}$) functions such that for each i the set $C(f_i)$ is dense in X. Then $\bigcap_{i=1}^{n} C(f_i)$ is dense in X.

Proof: Each set $C(f_i)$ is a dense G_δ subset of X so by Theorem 2(b) and induction on n we have $\bigcap_{i=1}^{n} C(f_i)$ is dense. □

Note that if X is a Baire space then the conclusion of Theorem 4 holds even for a countably infinite family of functions, but Example 1 shows that this is not the case if X is merely strongly Volterra.

We cannot relax "strongly Volterra" in this theorem to "Volterra" and then merely require that $\bigcap_{i=1}^{n} C(f_i)$ is nonempty when $n > 2$, even when X is a T_1 space, as the following example shows.

EXAMPLE 3: Let $A = \{(x, y) \in \mathbb{R}^2 / y \geq 0\}$ and for each $r \geq 0$ let $A_r = \{(x, y) \in \mathbb{R}^2 / y + r > 0\}$. Define B and B_r to be the sets obtained by rotating A and A_r 120° about (0, 0) in the anticlockwise direction and C and C_r by a similar rotation in the clockwise direction. Define the sets D and E by

$$D = (A_0 \cap B_0) \cup (B_0 \cap C_0) \cup (C_0 \cap A_0)$$

and

$$E = (A_0 - B - C) \cup (B_0 - C - A) \cup (C_0 - A - B).$$

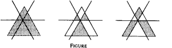
Let

$$\mathcal{B}_1 = \{A_r \cap B_s \cap C_t - F / r, s, t > 0 \quad \text{and} \quad F \subset \mathbb{R}^2 \quad \text{is finite}\}$$

$$\mathcal{B}_2 = \{A_r \cap B_s \cap C_t \cap D - F / r, s, t > 0 \quad \text{and} \quad F \subset \mathbb{R}^2 \quad \text{is finite}\}$$

$$\mathcal{B}_3 = \{A_r \cap B_s \cap C_t \cap E - F / r, s, t > 0 \quad \text{and} \quad F \subset \mathbb{R}^2 \quad \text{is finite}\}.$$

Setting $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ it may be verified that $\mathcal{B}$ is closed under finite intersections and covers $\mathbb{R}^2$ so forms a basis for a topology, $\mathcal{T}$, on $\mathbb{R}^2$.



FIGURE

The figure shows the three types of basic open subsets of $(\mathbb{R}^2, \mathcal{T})$. Clearly, $\mathcal{T}$ is T_1 as it contains all cofinite sets. So also are the sets A, B, and C are dense G_δ sets. So also are the sets $A \cap B_r \cap C_s - F$ for $r, s > 0$ and F countable, as well as those obtained by permuting A, B, and C. However, $A \cap B$ is not dense as $A \cap B \cap E = \varnothing$. Thus, the sets $A \cap B_r \cap C_s - F$ and their (A, B, C)-permutations are the "smallest" dense G_δ sets, where $r, s > 0$ and F is countable. Note that the intersection of any two of these sets is of the form $A \cap B \cap C_r - F$, or an (A, B, C)-permutation, where $r > 0$ and F is countable, so $(\mathbb{R}^2, \mathcal{T})$ is a Volterra space.

Consider the three subsets $A - \{(0, 0)\}$, B and C. These are dense G_δ subsets so by the lemma there are functions $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ for which $C(f) = A - \{(0, 0)\}$, $C(g) = B$ and $C(h) = C$. As $C(f) \cap C(g) \cap C(h) = \varnothing$, it is seen that these functions violate the Volterra analogue of Theorem 4.

REFERENCES

1. GAULD, D.B. 1993. Did the young Volterra know about Cantor? Math. Magazine. 66: 246–247.
2. GAULD, D.B., & Z. PIOTROWSKI. 1993. On Volterra spaces. Far East J. Math. Sci. 1: 209–214.
3. VOLTERRA, V. 1881. Alcune osservasioni sulle funzioni punteggiate discontinue. G.di Mat. 19: 76–86; 1954 Opera Matematiche di Vito Volterra. Accad. Naz. dei Lincei, Roma. 1: 7–15.