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# SEPARATE VERSUS JOINT CONTINUITY — AN UPDATE *

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This survey is an update of selected, major results in the rapidly growing field of separate and joint continuity, compare [Pt1]. The emphasis here is placed on the new findings, including historical ones, since the mid-80's.

**The genesis of separate versus joint continuity.** Way back in the year 1821, it was Augustin-Louis Cauchy who wrote incorrectly in his famous "Cours d'Analyse" [Ca] that a function of several variables which is continuous in each one separately, is continuous as a function of all the variables. Further studies of continuity were conducted by K. Weierstrass; his lectures at the University of Berlin (1858–59) were written down, only in 1861 by A. Schwarz [Kl], p. 177 and 952. The first written account of a counterexample to Cauchy's statement can be traced however to 1873 edition of J. Thomae "Abriss einer Theorie der complexen Funktionen", [Th*]. Here is what he states:

"...One can easily commit the error (as Mr. E. Heine has pointed out) of considering a function of two variables to be continuous if for every point, $abs[\omega(y \pm \zeta\delta, z) - \omega(y, z)]$ and $abs[\omega(y, z \pm \zeta'\delta) - \omega(y, z)]$ converge toward zero with decreasing $\delta$. However, that would mean, for example, that the function $\omega(y, z) = \sin 4\mathrm{arctg}\frac{y}{z}$, which we define for $z = 0$ by assuming that it is equal to zero along the entire $y$ axis (in the $y, z$ plane), would be continuous within the circle $y^2 + z^2 = 1$..."

The above Heine's example has the following form in cylindrical coordinates:

$$F(r, \theta) = -\sin 4\theta.$$

It can be easily noticed that the following simplification:

$$f(r, \theta) = \sin 2\theta$$

is also a counterexample to Cauchy's statement.

Going back to rectangular system the just written function has the following representation:

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } x^2 + y^2 \neq 0, \\ 0, & \text{if } x^2 + y^2 = 0. \end{cases}$$

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But this is the well-known example provided for the first time in 1884 by G. Peano [Ge*]! The real breakthrough came in 1899 with the appearance of the classical work of R. Baire [Ba*], who pioneered the method of category and put an end to the privilleged status of continuity. For more information on the genesis of separate versus joint continuity an interested reader is directed to [Pt2]. Also, in what follows, separately continuous function will be denoted by *SC* function.

**2. Findings of R. Baire and the main problems of separate and joint continuity.** Baire made significant progress in this field; he was the first to show that:

(*) For an arbitrary *SC* function $f\colon [0,1]\times[0,1]\to\mathbb{R}$, there is a residual set $S$ of $[0,1]$ such that $S\times[0,1]$ is made entirely of points of continuity of $f$.

Another property of all *SC* functions $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ can be found in [Ba*], p. 75 and was given by V. Voltera:

(**) For every point $(x_0,y_0)\in\mathbb{R}\times\mathbb{R}$, for every disk $K$ centered at $(x_0,y_0)$ and for every $\varepsilon>0$, there is a disk $K_1$ contained in $K$ such that $|f(x,y)-f(x_0,y_0)|<\varepsilon$, for every $(x,y)$ from $K_1$.

This property was later termed *quasi-continuity*.

Yet another property of all real-valued *SC* functions from the plane that was found by R. Baire is:

(* * *) Every *SC* function $f\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is of the first class (of Baire).

Based on Baire's (*), (**) and (* * *) and their refinements (more to come!) we are ready to formulate the main three problems of, what is called, *separate versus joint continuity*.

Let $X$ and $Y$ be "nice" spaces (e.g. Polish, or second countable Baire), let $M$ be metric and let $f\colon X\times Y\to M$ be *SC*.

**I. Existence Problem.** Find the set $C(f)$ of point of continuity of $f$; this set is usually a dense $G_\delta$ subset of $X\times Y$.

**"Fiber version"** — it is similar to the above, except for now we look for $C(f)$ in $\{x\}\times Y$, for every fixed $x$ in $X$.

**II. Characterization Problem.** Characterize $C(f)$ as a subset of $X\times Y$. Usually, it is the complement of an $F_\sigma$ set contained in the product of two sets of first category.

**III. Uniformization Problem.** Find a "uniform", "thick" subset $A$ of $X$ such that $A\times Y$ is contained in $C(f)$. Again, usually, the set $A$ is a dense $G_\delta$ subset of $X$.

These Problems originally came out of real analysis, have topological formulations and can be solved using usually methods of functional analysis.

Let us now outline some answers to the above three major problems.

**3. Separate continuity and Baire classification of functions.** Let us start, by showing, following Lebesgue, that all real-valued *SC* functions from the plane are of the first class of Baire. In fact, let $f$ be such a $SC$ function. For each natural number $n$, draw vertical lines in the plane, each at distance $\frac{1}{n}$ from its left and right neighbours. Define $f_n(x,y)$ to be $f(x,y)$ on the union of these lines and determine $f_n(x,y)$ on the rest of the plane by linear interpolation in the variable $x$. Since $f$ is continuous of $y$, for each fixed $x$, each $f_n$ is a continuous function on $\mathbb{R}^2$. Since $f$ is continuous function of $x$, for each fixed $y$, $\lim_{n\to\infty} f_n(x,y) = f(x,y)$, for all $(x,y) \in \mathbb{R}^2$. So, we have obtained the function $f$ as the limit of a sequence of continuous functions.

A much more abstract version of the above theorem, for real-valued $SC$ functions defined on products of any two *metric spaces* is known as *Baire-Kuratowski-Montgomery theorem*. By yet another theorem of R. Baire, every Baire 1 function is pointwise discontinuous, i.e. its set of points of discontinuity is of the first category. Now, since the domain (the plane) is a Baire space, the set $C(f)$, being the complement of a first category is dense, in fact $G_\delta$ subset, of the plane. So, as a dense $G_\delta$ subset of $\mathbb{R}^2$, the set $C(f)$ of points of continuity of *any* real-valued $SC$ function from the plane is *uncountable* (!) The same conclusion holds for all real-valued $SC$ functions defined on a Baire space $X \times Y$, where $X$ and $Y$ are metric. Observe that the above result answers our Existence Problem — see Section 2.

Coming back to the question of when are real-valued $SC$ functions of $1^{st}$ class, let us mention only W. Rudin's [Ru] theorem which says that it is so, if the pointwise compact subsets of $C(X)$ are metrizable and $Y$ is compact, see also W. Moran's [Mo*], G. Vera [Ve] and the just published result of M. Henriksen and R.G. Woods [HW].

Let us now derive yet another corollary from the classical Baire's result that all real-valued $SC$ functions are of $1^{st}$ class. Namely, an estimate of the cardinality of the class of $SC$ functions. Since there are $c^\omega = c$ many sequences having terms from a set of cardinality $c$, there are $c$ many real-valued $SC$ functions from the plane. Other estimates of the cardinality of the class of $SC$ functions will be given in Section 7.

There are, however, a few shortcomings of Baire classification when dealing with $C(f)$ of $SC$ functions. Since Baire 1 functions are defined as *limits of sequences of real-valued* continuous functions, this condition alone severly restricts possible applications. Perhaps we could use Lebesgue-Hausdorff theorem on the equivalence of Borel 1 and Baire 1 classes and ask which $SC$ functions are $F_\sigma$ measurable, for recent results on Lebesgue-Hausdorff theorem see [Ha] and [Fo]. Also, there is an easy example of a separately Baire 1 function $f\colon [0,1] \times [0,1] \to \mathbb{R}$ such that $C(f) = \emptyset$. This example shows a major weakness of Baire classification while searching for $C(f)$ of $SC$ functions.

Concluding this section let us mention that quasi-continuity is a better tool when studying continuity points of $SC$ functions with non-metrizable ranges; it will be examined next.

## 4. Quasi-continuity of separate continuity.

Observe that one really does not need two dimensions in the domain of $f$ in the condition (**) of Baire; further replacing disks with open sets, appropriately, we obtain the following topological version of (**):

(QC) Given $f: X \to Y$; we say that $f$ is *quasi-continuous* if for every $x$ from $X$, for every open set $U$ containing $x$ and every open set $V$ containing $f(x)$, there is a nonempty open set $U'$, with $U' \subset U$ and such that $f(U') \subset V$.

In the 60's, quasi-continuity was re-invented by Z. Frolík [Fr] while studying the invariance of Baire spaces under functions, in the 90's, quasi-continuity was defined once again, this time when dealing with some generalizations of Michael's selection theorem [BG].

Baire findings on quasi-continuity of separately continuous functions were refined by S. Kempisty [Kp*], N.F.G. Martin [Mt*], T. Neubrun [Ne] and Z. Piotrowski [Pt3*]; apparently if $X$ is Baire, $Y$ is second countable and $Z$ is regular, then separate quasi-continuity (hence: separate continuity) implies quasi-continuity of $f: X \times Y \to Z$. Since metrizability or countability of basis of either $X, Y$ or $Z$ severly restricts applications of separate versus joint quasi-continuity into functional analysis or topological algebra, the following result is of interest: [PS1]. Let $X$ and $Y$ be Čech-complete and let $Z$ be completely regular, then separate continuity of $f: X \times Y \to Z$ implies quasi-continuity, see also [Tr].

Also, for a long time it was known that every quasi-continuous function defined on a Baire space $X$ and having values in *either* a metric space *or* a second countable space, has a dense $G_\delta$ subset of the set $C(f)$ of points of continuity [Ne].

For a couple of years I knew that the class of all spaces $Y$ that have a countable pseudo-base (even open-hereditary) and are simultaneously hereditarily Lindelöf is "*too large*"; take $X$ to be the reals with the usual topology and $Y$ to be the reals with the Sorgenfrey topology. Then the identity function $f: X \to Y$, i.e. $f(x) = x$, is quasi-continuous which is continuous at *no point*. This is why I asked [Pt3] for which large class of spaces $Y$, every quasi-continuous function $f: X \to Y$, defined on a Baire space $X$ has at least one point of continuity.

Recently [KKM], P. S. Kenderov, I. S. Kortezov and W. B. Moors answered my question providing a nice characterization which uses the notion of a fragmentable space. The latter notion, defined by [JR], was intensively studied by N. K. Ribarska [Rb1] and [Rb2], who gave a necessary and sufficient condition for a space to be a fragmentable one and proved that a fragmentable compact Hausdorff space is fragmented by some complete metric, see also [Rb3] and [Ng2].

## 5. Characterization of the set $C(f)$.

We shall start describing $C(f)$ by exhibiting results on how "large" (in various senses) can the set of Discontinuity $D(f)$ be, given a $SC$ function.

It is an easy exercise that $D(f)$ can be *countably dense* in $[0,1] \times [0,1]$. In fact, let $D = \{(x_i, y_i): i \in N\}$ be any dense and countable subset of $[0,1] \times [0,1]$, then $f$ defined by:

$$f(x, y) = \sum_{n=1}^{\infty} \frac{f_n(x, y)}{2^n}, \quad \text{for each } n,$$

$$f_n(x,y) = \begin{cases} \dfrac{2(x-x_n)(y-y_n)}{(x-x_n)^2+(y-y_n)^2}, & \text{if } (x,y) \neq (x_n,y_n) \\ 0, & \text{otherwise,} \end{cases}$$

is an $SC$ function with $D = D(f)$.

Using a family of Cantor sets in an interesting and clever process of densifying the set $D(f)$, G.C. Young and W.H. Young [YY*] showed that there is a function $f: [0,1] \times [0,1] \to \mathbb{R}$ which is continuous with respect to every straight line whose $D(f)$ is *uncountably* dense, i.e., it has uncountably many points of discontinuity in every nonempty open set contained in the unit square. T. Tolstoff [To*] showed that there is an $SC$ function $f: \mathbb{R}^2 \to \mathbb{R}$ whose set $D(f)$ has a *positive Lebesgue measure*. Let as note that H. Lebesgue [Lb*] knew already that a function $f: [0,1] \times [0,1] \to \mathbb{R}$ can be continuous along *every analytic curve* (hence, $SC$) through $(x_0, y_0)$ and still discontinuous at $(x_0, y_0)$, e.g., define $f$ as follows:

$$f(x,y) = \begin{cases} 1, & \text{if } y = e^{-\frac{1}{x^2}}, \text{ except for } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

T. Körner in yet unpublished paper [Kö], constructed a function $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is continuous along every analytic curve whose set $D(f)$ has *Hausdorff measure* 1 *densely*, i.e., $D(f)$ has Hausdorff measure 1, when intersected with every non-empty open subset of the plane.

It follows from Baire's (*), Section 2 that the set $D(f)$, the set of discontinuity of a $SC$ function $f: [0,1] \times [0,1] \to \mathbb{R}$ is an $F_\sigma$ (as the complement of $C(f)$, being a $G_\delta$) that is contained in the *product of two sets of first category*. What was missing was the converse to the above result. It came, in 1944, done by R. Kersher [Kr*].

Namely: Let $S \subset \mathbb{R}^2$. Then $S$ is $D(f)$ of a certain $SC$ function $f: \mathbb{R}^2 \to \mathbb{R}$ iff $S$ is an $F_\sigma$ contained in the product of two sets of first category. This characterization holds also for domains being products of compact metric spaces [BN*]. Answering one of my earlier questions, V. K. Maslyuchenko, V. V. Mykhaylyuk and O. V. Sobchuk proved in 1992, see [MMS] that the above characterization of $D(f)$ for $SC$ functions is no longer true if $X$ and $Y$ are arbitrary compact Hausdorff spaces.

The following spectacular question of M. Talagrand [Ta2*] is, however, still open:

Let $X$ be Baire, $Y$ be compact (Hausdorff) and let $f: X \times Y \to \mathbb{R}$ be any $SC$ function. Is $C(f) \neq \emptyset$?

Carefull reader has surely noticed that separate continuity was replaced, at times, in this section, by some other conditions, e.g., Young-Young linear continuity. More precisely, function $f: \mathbb{R}^2 \to \mathbb{R}$ is called *linearly continuous* if it is continuous with respect to every straight line in the plane. In the past linear continuity received much less attention than separate continuity. Still, following M. Slobodnik [Sl*], the set $D(f)$ for an arbitrary linearly continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ is *always of measure* 0, so, as an $F_\sigma$ it is *of first category*. A complete characterization of $D(f)$ for an arbitrary linearly continuous function is still unknown.

**6. Namioka-type theorems.** This section is devoted to various answers to our Uniformization Problem of Section 2. This is, by far, the most celebrated part of the topic of separate versus joint continuity.

Observe that the first result of this type is due to R. Baire himself [Ba*], see (*) of Section 2. The set $A$ mentioned in the formulation of the Uniformization Problem is a dense $G_\delta$; Baire showed that this set is residual. However, if $X$ is a Baire space, it follows from the Baire Category Theorem that every dense $G_\delta$ subset of $X$ is residual and every residual subset of $X$ *contains* a dense $G_\delta$ set. By the way, it was shown (J. Saint Raymond [SR2*]) that if $X$ is completely regular, $Y$ is compact Hausdorff, $M$ is metric and if for every $SC$ function $f\colon X \times Y \to M$ there is a dense $G_\delta$ subset $A$ with $A \times Y \subset C(f)$, then $X$ is, in fact, Baire.

Following H. Hahn [Hh2*] a space $S$ is called an *absolute* $G_\delta$ or a *Young space*, if it is a $G_\delta$ subspace in every metric space $X$ that contains $S$. In view of the remetrization theorem of Alexandroff, Young spaces coincide with topologically complete and are labeled now Čech-complete. Hahn, who devoted to separate vs. joint continuity fourteen pages of §39 of his "Reelle Funktionen" showed the following result [Hh2*], §39:

Let $X$ be a metric Young space, $Y$ be compact metric, and let $f\colon X \times Y \to \mathbb{R}$ be $SC$, then there is a residual subset $A$ of $X$ such that $A \times Y \subset C(f)$.

The very first answer to our Uniformization Problem where no assumptions of either *metrizability* or *countability of basis* are made upon either $X$ or $Y$ came in 1971 in a paper by S. L. Troyanski [Ty*] on renorming a Banach space that is generated by a weak-compact set. Troyanski's renorming theorem implies the following result: Let $X$ and $Y$ be compact Hausdorff spaces and let $f\colon X \times Y \to \mathbb{R}$ be a *bounded* $SC$ function, then there are dense $G_\delta$ sets $A$ and $B$ of $X$ and $Y$, respectively such that $A \times Y \cup X \times B \subset C(f)$.

Finally, in 1974, I. Namioka showed [Na*]:

Let $X$ be regular, strongly countably complete, $Y$ be locally compact and $\sigma$-compact and let $Z$ be pseudo-metric. Then for any $SC$ function $f\colon X \times Y \to Z$ there is a dense $G_\delta$ subset $A \subset X$, such that $A \times Y \subset C(f)$.

Notice that Namioka's theorem obviously generalizes the result of Troyanski, whereas Hahn's theorem can be viewed as a *metric version* of Namioka theorem.

The original proof of Namioka theorem starts with an interesting reduction to the case when $Y$ is *compact*. Next, using purely topological methods, such as, e.g. Arhangel'skii-Frolík covering theorem, or Kuratowski's theorem on closed projections, Namioka shows that, given the set $O_\varepsilon$ being the union of all open subsets $O$ of $X \times Y$ such that diam $f(O) \le \varepsilon$, the set $A_\varepsilon = \{x\colon \{x\} \times Y \subset O_\varepsilon\}$ is dense in $X$. Generalizations of Namioka theorem e.g. [Cr1*], or [Ta2*] use, as $X$, various spaces defined by an appropriate version of Banach-Mazur game; also methods of functional analysis, especially function spaces (Mazur and Eberlein theorems [To]) are frequently applied.

Result of I. Namioka initiated, no doubt, the renaissance of the topic of separate versus joint continuity. The problem was: How "far" can we go, i.e. what types of spaces can be assumed as $X$ or $Y$? In particular, the questions were:

(a) [AO*], [Cr1*]: Does the conclusion of Namioka theorem hold if *both* $X$ and $Y$ are arbitrary *complete* metric spaces?

(b) [Na*]: Does the conclusion of Namioka theorem hold if $X$ is assumed to be a Baire space?

The answer to *both* (a) and (b) is "no", and was shown by J. B. Brown [Pt1] and M. Talagrand [Ta2*], respectively.

So, "the candidates" for $X$ in Namioka theorem are various "almost" Baire, game-defined spaces, while "the candidates" for $Y$ are various compact-like spaces.

Following J. P. R. Christensen we say that a Hausdorff space $X$ is *Namioka* or that $X$ has *the Namioka property* $\mathcal{N}$, if for any compact space $Y$, any metric space $Z$, and any $SC$ function $f: X \times Y \to Z$ the conclusion of Namioka theorem holds. J. Saint Raymond [SR2*] shows the following important results:

(i) Separable Baire spaces are Namioka

(ii) Completely regular Namioka spaces are Baire

(iii) In the class of *metric* spaces:

$X$ is Namioka if and only if it is Baire.

Although M. Talagrand exhibited an $\alpha$-favorable space $X$ (hence, Baire) which does *not* have the Namioka property $\mathcal{N}$, still there are many natural game-defined spaces that *are* Namioka.

Clearly, by Namioka theorem, Čech-complete spaces are Namioka, as well as, $\sigma$-well $\alpha$-favorable [Cr1*], $\sigma$-$\beta$ defavorable [SR2*], $\tau$-$\beta$ defavorable [Ta2*] and [Db1].

Following G. Debs [Db1] we say that a compact space $Y$ is *co-Namioka* or has *the Namioka property* $\mathcal{N}^*$, if for every Baire space $X$ and for every $SC$ function $f: X \times Y \to \mathbb{R}$, the conclusion of Namioka theorem holds. It was shown that $\mathcal{N}^*$ contains many compact-like spaces appearing in functional analysis; among them are Eberlein compact [Dv], Corson compact [Db2], Valdivia compact [DG], and, more generally, all compact spaces $Y$ such that $C_p(Y)$ is $\sigma$-fragmentable [JNR]. As it was shown by R. Deville [Dv], $\beta N \notin \mathcal{N}^*$. Recently A. Bouziad [Bo1] showed that $\mathcal{N}^*$ contains all scattered compact spaces which are hereditarily submetacompact. Many interesting results in this field have been obtained by I. Namioka and R. Pol, see Section 8.

Also permanence properties of both Namioka and co-Namioka spaces have been studied; it is known that the class $\mathcal{N}^*$ is closed under continuous images, arbitrary products [Bo2] and countable unions [Hd1]. In view of Saint Raymond's characterization of metric Namioka spaces as Baire spaces, we conclude that perfect, continuous functions, in general, do *not* preserve Namioka spaces; also the product of two, even metric, Namioka space *need not* be Namioka. The reals with the Sorgenfrey topology plays a very interesting role in this topic. Despite the fact that it is an $\alpha$-favorable space, *it is* Namioka (see [Pt1]); recall that Talagrand's Baire space which is *not* Namioka is $\alpha$-favorable. On the other hand, if $X$ is Baire, $Y$ is the Sorgenfrey line and $f: X \times Y \to \mathbb{R}$ is an arbitrary $SC$ function, then the conclusion of Namioka-type theorem *fails* (V.K. Maslyuchenko, private communication).

There is yet another group of results related to Namioka theorem. It has been shown that if $Y$ *is* second countable, then the conclusion of Namioka theorem is true, if $X$ is an *arbitrary* Baire space. This result was obtained independently by P.S. Kenderov [Ke2*] who used Fort's category theorem and properties of multifunctions, J. Calbrix and J.P. Troallic [CT*] who used method of function spaces, and J.B. Brown [Pt4], who used only Baire Category Theorem and the properties of metric. The just mentioned result of Maslyuchenko shows the necessity of second countability of $Y$ in the above results of Kenderov, Calbrix-Troallic and Brown; countable pseudo-base for $Y$ does not suffice.

Also multi-valued versions of Namioka theorem have been studied, see [La] and [Ch].

## 7. Other related topics.

**7.1. Sierpiński's theorem on the determination of $SC$ functions.** In 1932, W. Sierpiński [Si] showed that any real-valued, $SC$ function on $\mathbb{R}^n$ is uniquely determined by its values on *any* dense subset of the domain space; in other words, if two continuous functions agree on a dense subset of $X$, then they agree throughout $X$.

Sierpiński's result has been proven again in R. A. McCoy and [To*] and generalized by C. Goffman, C. J. Neugebauer, W. W. Comfort, and quite recently by E. J. Wingler and myself [PW1]. In fact, [PW1], Structural Lemma, p. 17 we have: Assume that every $SC$ function from the product $X = X_1 \times X_2 \times \cdots \times X_n$ into a completely regular space $Z$ has the following property:

$$\text{(FC)} \qquad \forall \underset{\text{open}}{V} \subset Z\colon f^{-1}(V) \neq \emptyset \Rightarrow \operatorname{Int} f^{-1}(V) \neq \emptyset.$$

Then any $SC$ from $X$ into $Z$ is determined by its values on any dense subset of the domain space.

It is so, e.g., if $X_1 = X_2 = \cdots = X_n$ are Baire, second countable spaces, the resulting $f$ is then quasi-continuous and, as such, satisfies the condition (FC) of feeble continuity, compare also Section 4.

**7.2. Conditions implying continuity of a $SC$ function.**

a) *Young's monotonicity theorem*

W.H. Young [Yo*] gave one of the first result of this type, namely:

*$SC$ function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that are monotone (i.e. increasing or decreasing) in one variable are continuous.*

This result has been re-discovered (e.g., [DK*]) or is not being given credit to anybody (folklore) in real analysis textbooks.

b) *Lusin's restriction and Rosenthal's convexity theorems*

N. Lusin [Lu*] proved that a function $f\colon [a,b] \times [c,d] \to \mathbb{R}$ is continuous if and only if its restriction to the graph of *each* continuous function $g\colon [a,b] \to [c,d]$ and $h\colon [c,d] \to [a,b]$ is continuous.

What follows is one of two generalizations of Lusin's result that can be found in J. P. Dalbec's ([Da]; p. 671):

Let $X$ be completely regular. Let $Y$ be first countable and locally path-connected let $Z$ be topological space. Suppose that $X \times Y$ is sequential, that the function $g: X \times Y \to Z$ has continuous $x$-sections, and that for any continuous function $f: X \to Y$, the function $g_f$, defined by $g_f(x) = g(x, f(x))$, is continuous. Then $g$ is also continuous.

Recall (Section 5) that H. Lebesgue showed that a function $f: \mathbf{R}^2 \to \mathbf{R}$ can be discontinuous at $(x_0, y_0)$ even though $f$ is continuous along every analytic curve through $(x_0, y_0)$. However, A. Rosenthal [Rs*] showed that if $f$ is continuous along every *convex* curve which is at least once differentiable, then $f$ is continuous. Yet, [Rs*], $f$ can be continuous along every curve through $(x_0, y_0)$ which is *twice differentiable* without being continuous at $(x_0, y_0)$.

c) *Closed Graph Property*

Is a closed graph $SC$ $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ continuous? The answer to this question is "yes" (see [PW2]) and it came only in 1990. So, what specific conditions upon spaces $X, Y$ and $Z$ guarantee that a closed graph $SC$ function $f: X \times Y \to Z$ is continuous? Here is the answer; [PW2], Theorem 1:

Let $X$ and $Y$ be topological spaces with $Y$ locally connected. Let $Z$ be locally compact and suppose $f: X \times Y \to Z$ has continuous $y$-sections and connected $x$-sections. if $f$ has a closed graph, then $f$ is continuous.

d) *Pettis' near continuity*

As it was shown by B. J. Pettis the lack of linearity of an operator in Closed Graph Theorem may be compensated, in general topological case (the considered spaces are not necessarily assumed to the linear) by the following conditon of *near continuity of $f$* at every $x$:

(NC) $$\underset{\text{open}}{\forall V}: f(x) \in V \Rightarrow x \in \operatorname{Int} \overline{f^{-1}(V)},$$

that is, e.g. every closed graph, nearly continuous function $f: X \to Y$ between any two complete metric spaces $X$ and $Y$ is continuous, see also [PS] for further generalizations.

Since quasi-continuity and near continuity constitute a *decomposition* of continuity ([Ne]) i.e. a function that is simultaneously nearly continuous and quasi-continuous, is continuous, so that for significantly large class of spaces, separate continuity implies quasi-continuity we have the following result; compare Section 4:

Assume $Z$ is completely regular. If

(i) $X$ is Baire and $Y$ is second countable *or*

(ii) both $X$ and $Y$ are Čech-complete.

Then every nearly continuous $SC$ function $f: X \times Y \to Z$ is continuous.

**7.3. Cluster sets techniques answer the Uniformization Problem.** For definitions and earlier results, the authors is refered to [P1]. The new result since the mid-80's that is very interesting is Nagamizu theorem [Ng1]:

Let $Y$ be second countable, $Z$ be compact metrizable, let $f: X \times Y \to Z$ has all $x$-sections continuous, $x \in X$, and has all $y$-sections quasi-continuous, $y \in E$, where $E$ is dense in $Y$. Then there is a residual subset $A \subset X$, such that $A \times Y \subset C(f)$.

Observe that $y$-sections here are assumed to be quasi-continuous; recall [Ne] that there are quasi-continuous functions $f: \mathbf{R} \to \mathbf{R}$ which are even not Lebesgue measurable.

**7.4. Invariants of separate continuity.** As we noticed (Section 4) for "nice" classes of spaces (e.g., $X$-Baire, $Y$-second countable or both $X$ and $Y$-Čech-complete with $Z$ being completely regular, in both cases) separate continuity implies quasi-continuity. Now, quasi-continuous functions as feebly continuous preserve e.g. separability, see [Ne] for more properties.

It is easy to see that separately continuous functions do not preserve, in general, connected sets — take the set $\{(x, y): y = x\}$ and use the standard example of a separately continuous function which is not continuous at $(0, 0)$. However, R.A. Mimna [Mi] showed that if $f: X \times Y \to Z$ is a function with connected $x$-sections and connected $y$-sections, where $X$ and $Y$ are locally connected Hausdorff spaces, and $Z$ is Hausdorff, then the image of every connected *open* set in $X \times Y$ is connected in $Z$.

**7.5. Cardinality of $SC$ function.** As mentioned in Section 3 there are $c$ many real-valued $SC$ functions from the plane. It has been shown [PRS], that if $d(X), d(Y) \leq \kappa$ for some infinite cardinal $\kappa$, then there are at most $2^{\kappa}$ $SC$ functions $f: X \times Y \to \mathbf{R}$, as usual, $d(X)$ denotes the density of $X$. If there are cellular families (i.e. families of pairwise disjoint nonempty open subsets of a space) in $X$ and $Y$ of cardinality $c(X)$, then M. Henriksen and R.G. Woods [HW] showed that the cardinality of all $SC$ functions is greater or equal to $2^{c(X)}$, where $X$ and $Y$ are completely regular spaces with $c(X) \leq c(Y)$.

There are spaces $X$ and $Y$ such that the cardinality of the class of all $SC$ functions $f: X \times Y \to \mathbf{R}$ is *strictly greater* than the cardinality of the class of all continuous functions from $X \times Y$ into $\mathbf{R}$; in fact take $X$ to be *any* $T_1$ space with $\kappa \geq \omega$ isolated points. Then the cardinality of all real-valued $SC$ functions $f: X \times X \to \mathbf{R}$ is at least $2^{\kappa}$, see also [HW] for a corresponding result in terms of cellular families.

**7.6. Topology of separate continuity.** Following works of J. Novak; C. J. Knight, W. Moran, J. S. Pym; A. V. Arhangel'skii; and S. Popvassilev, the recent work [HW] (look there for the references of just mentioned articles) brings many deep results on the relations between the product topology $\tau$, the cross topology $\gamma$ (where a subset of $X \times Y$ is open in the cross topology if its intersection with each vertical fiber and each horizontal fiber is open in the subspace topology induced on the fibers by $\tau$), and the topology $\sigma$ of separate continuity, which is the weak topology on $X \times Y$ generated by the family of all $SC$ functions, e.g., it is shown that if $X$ is a compact metrizable space without isolated point, then the topology $\sigma$ of separate continuity for all $SC$ functions $f: X \times X \to \mathbb{R}$ is not regular.

**8. Applications.** As in previous sections the reader is asked to consult the corresponding part of author's 1985 "Separate and joint continuity" [Pt1], first.

There is a natural link between Namioka or co-Namioka spaces and special maps into function spaces. In order to exhibit this connection let us give appropriate definitions first. For a compact space $K$, let $C(K)$ denote the Banach space of all real-valued continuous functions on $K$ with the supremum norm. Besides the usual norm-topology $\tau_n$ on $C(K)$, we also consider the topology $\tau_p$ of pointwise convergence on $K$. Following G. Debs, a space $X$ is Namioka, if for every $\tau_p$-continuous function $f: X \to C(K)$, the set of $\tau_n$-continuity points is dense in $X$. Notice that the set of continuity points of any mapping from a topological space into a metrizable space is always $G_\delta$, so the word "dense" can be replaced by "dense $G_\delta$". Further, a compact space $K$ has the property $\mathcal{N}^*$ of co-Namioka, if for each $\tau_p$-continuous function $f$ from a Baire space $B$ into $C(K)$, the set of $\tau_n$-continuity points is dense in $B$.

Many results obtained in the 90's pertain to detailed studies of $\tau_p$-continuous functions $f: X \to C(K)$, especially with regard to fragmentability or $\sigma$-fragmentability of subsets of $C(K)$, compare P. S. Kenderov and W. B. Moors [KM] or I. Namioka and R. Pol [NP1], where a lot of research has been done in the area of property $\Sigma$ ($K$ is said to have property $\Sigma$ if $(C(K), \tau_p)$ is $\sigma$-fragmented by the metric of the norm). In [NP2], assuming that there exists in the unit interval $[0,1]$ a coanalytic set of cardinality continuum without any perfect subsets Namioka and Pol, constructed a scattered co-Namioka space such that $C(K)$ does not admit a Kadeč norm that is equivalent to the supremum norm; this result answered some important questions by Deville, Godefroy [DG] and R. Haydon [Hd2].

Let us now turn our interest to topological groups. A semitopological (resp. partopological) group is a group endowed with a topology for which the product is separately (resp. jointly) continuous. In 1957, R. Ellis [El1*], [El2*] showed that every locally compact, semi-topological group is topological. This answered a question posed by A. D. Wallace. In 1982 N. Brand proved that every Čech-complete paratopological group is topological.

The final question whether every Čech-complete (even Čech-analytic) semitopological group is topological was answered affirmatively in 1996, by A. Bouziad [Bo2]; quasi-continuity is widely used there. Observe that this is a quite sharp estimate, since taking the inverse in the Sorgenfrey line, an $\alpha$-favorable space, is *not* continuous. Also E. A. Reznichenko [Re1] announced without proof the above result of Bouziad. In [Re2], it is shown that every completely regular pseudocompact paratopological group is topological and every completely regular countably compact semitopological group is topological.

Finishing, let us mention that it was Troyanski's renorming theorem [Ty*] that let R. Deville and G. Godefroy [DG] prove that Valdivia compact spaces are co-Namioka. In return Lemma II.3 of their result implies the following renorming result of G. Alexandrov [Al]:

Let $G$ be a compact topological group. There exists an equivalent norm on $C(G)$ which is locally uniformly rotund and translation invariant.

This phenomenon illustrates more and more frequent situation when a new result in separate versus joint continuity is obtained via "deep" new result in functional analysis.

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## ЧАСТИЧНА НЕПРЕКЪСНАТОСТ И СЪВМЕСТНА НЕПРЕКЪСНАТОСТ

**Збигнев Пиотровски**

Този обзор е продължение на обзора от статията [Pt1] и съдържа по-важните резултати относно съвместна непрекъснатост на частично непрекъснати функции. Ударението е поставено върху новите находки (включително и относно историята на изследванията) след средата на 80-те години.