SEPARATE AND JOINT CONTINUITY IN BAIRE GROUPS

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Dedicated to Professor T. Šalát on the occasion of his 70th birthday

ABSTRACT. The celebrated theorem of R. E l l i s (1957) stating that separately continuous multiplication in a locally compact group is continuous has been generalized by J. L a w s o n (1984), G. H a n s e l and J. P. T r o a l l i c (1983) and D. H e l m e r (1981).

A. B o u z i a d (Topology Appl. (1993)) showed that the multiplication is continuous in a semitopological Baire group which is a paracompact p-space.

He also recently showed (Proc. Amer. Math. Soc (1996)) that the multiplication is continuous in semitopological Čech-complete groups. We shall prove (Theorem 5) that if X is a Baire, Moore semitopological group, then X is paratopological.

I. Introduction

Let us recall that a group G is called *semitopological* (resp. *paratopological*) if G is equipped with a topology in which the multiplication is separately (resp. jointly) continuous.

Further, a paratopological group G is called *topological* if the inversion in G is continuous.

This way we have two natural, very important questions:

(A) What is a topology on G such that every semitopological group with this topology is paratopological?

(B) What is a topology on G such that every paratopological group with this topology is topological?

Both questions (A) and (B) received a lot of attention in the past.

It was R. Ellis [10], [11] who showed that local compactness of G is an answer to both (A) and (B).

Answering (A), A. Bouziad [2] showed that every semitopological Baire group which is a paracompact p-space, is paratopological. Also Čech-completeness answers (A)—see [3].

The main purpose of this paper is to answer (A) for some other Baire groups.

The main result (Theorem 5) is a straight corollary from the previous theorems of this paper.

II. Some topological results

In this chapter we define some necessary notions used in the sequel, as well as prove Theorem 1. By a space we shall mean a topological space.

A function $f: X \to Y$ is called *quasi-continuous at a point* $x \in X$, if for every A and H, where $x \in A$ and $f(x) \in H$ we have:

$$A \cap \operatorname{Int} f^{-1}(H) \neq \emptyset.$$

A function $f: X \to Y$ is called *quasi-continuous* if it is quasi-continuous at every point of X. This condition defined by V. Volterra [1], p. 95 has been applied into separate versus joint continuity problems [14] or [31]; recently [8]; under the name "modified continuity" was used in a generalization of Michael's Selection Theorem. Let us mention here that there are no Lebesgue measurable quasi-continuous functions $f: \mathbb{R} \to \mathbb{R}$, and that quasi-continuous functions can be arbitrary class of Baire. [22].

Further, given X, Y and Z; a function $f: X \times Y \to Z$ is said to be *quasi-continuous with respect to the variable* x *at* $(p,q) \in X \times Y$ if for every neighborhood N of $f(p,q)$ and for every neighborhood $U \times V$ of (p,q) there exists a neighborhood of p, $U^1 \subset U$ and a nonempty open $V^1 \subset V$ such that for all $(x,y) \in U^1 \times V^1$ we have $f(x,y) \in N$. If f is quasi-continuous with respect to the variable x at every point of its domain, we say that f is *quasi-continuous with respect to* x. Let us recall

THEOREM 1. [24] Let X be first countable, Y be a Baire space, and Z be regular. If $f: X \times Y \to Z$ has all of its x-sections f_x quasi-continuous and all of its y-sections f_y continuous, then f is quasi-continuous with respect to x.

Remark 1. Quasi-continuity of the sections f_x in Theorem 1 cannot be weakened to the following one: inverse images of nonempty open subsets V of Y, if nonempty, have nonempty interiors. The reader will easily construct such a function $f: I^2 \to \mathbb{R}$.

III. Generalized oscillation function and quasi-continuity

First we need a couple of definitions. If $A \subset X$ and $\mathcal{U}$ is a collection of subsets of X, then the star of A with respect to $\mathcal{U}$, denoted by $\operatorname{st}(A, \mathcal{U})$ is defined as follows: $\operatorname{st}(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U}: U \cap A \neq \emptyset\}$.

For $x \in X$, we write $\operatorname{st}(x, \mathcal{U})$ instead of $\operatorname{st}(\{x\}, \mathcal{U})$.

A sequence $\{G_n\}$ of open covers of X is called a *development* of X if for each $x \in X$, the set $\{\operatorname{st}(x, G_n): n \in \mathbb{N}\}$ is a base at x.

A *developable space* is a space which has a development. Every metric space is developable; take the family B_ε of balls of diameter less than ε as a development.

A regular, developable space is called a *Moore space.*

Although some similar notions have been used earlier, see [6] or [17], the original idea of a generalized oscillation function Ω is due to A. Szymański (private communication), along with most of the comments preceeding Lemma 1, see below.

If $f: X \to Y$ is a function and $\mathcal{P}_n$ is an open cover of Y, then we set:
$\Omega_n(f, \mathcal{P}_n) = \{x \in X: \text{there is an open neighborhood } U \text{ of } x \text{ and a member } P_n \text{ of } \mathcal{P}_n \text{ such that } f(U) \subset P_n\}.$

If Y is a metric space and $\mathcal{P}_\varepsilon$ is the family of all balls of diameter less than ε, then $\Omega_\varepsilon(f, \mathcal{P}_\varepsilon)$ is the set of all points, where the oscillation of f is less than ε.

EXAMPLE 1. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Let $\varepsilon = 1$, then $\Omega_\varepsilon(f, \mathcal{P}_\varepsilon) = \mathbb{R} \setminus \{0\}$.

Remark 2. Let $\mathcal{P}_n$ be an open cover of Y. Then the set $\Omega_n(f, \mathcal{P}_n)$ is open.

LEMMA 1. Let X and Y be spaces and let $\{P_n\}$ be a development for Z. If $f: X \times Y \to Z$ is quasi-continuous with respect to x, then $\Omega_n(f, P_n)$ is dense in $\{x\} \times Y$, for every $x \in X$ and arbitrary but fixed n.

Proof. Let $x_0 \in X$ be arbitrary and let G be a nonempty open subset of Y. Further, let y_0 be an arbitrary element of Y. and let P_n be an element of the development $\{\mathcal{P}_n\}$ such that $f(x_0, y_0) \in P_n$.

By quasi-continuity of f with respect to x at (x_0, y_0), for every neighborhood N of $f(x_0, y_0)$, and every neighborhood $U \times V$ of (x_0, y_0), there is a neighborhood U' of x_0 with $U' \subset U$ and a nonempty open $V' \subset V$ such that for all $(x, y) \in U' \times V'$ we have $f(x, y) \in N$.

Since the sets N and V are arbitrary, we may assume $N = P_n$ and $V = G$, as defined above.

Observe that $U' \times V' \subset U \times G$ whereas $f(U' \times U') \subset P_n$. We have also $(U' \times V' \cap (\{x_0\} \times G)) = \{x_0\} \times V' \neq \emptyset$.

The above two facts imply the density of $\Omega_n(f, \mathcal{P}_n)$ in $\{x_0\} \times Y$, V' being a nonempty open subset of G, $n = 1, 2, \dots$.

Now, define $\Omega(f, \{\mathcal{P}_n\}) = \bigcap_{n=1}^{\infty} \Omega_n(f, \mathcal{P}_n)$. We have the following

LEMMA 2. Let $f: X \to Y$, where Y is a developable space with a development $\{\mathcal{P}_n\}$. Then $\Omega(f, \{\mathcal{P}_n\}) \subset C(f)$.

Proof. Let $x \in \Omega(f, \{\mathcal{P}_n\})$ and let V be an open set containing $f(x)$. Since Y is developable, there is an open countable base $\{V_n\}$ around $f(x_0)$.

Now, since $x \in \Omega(f, \{\mathcal{P}_n\})$, for every $n \in \mathbb{N}$, there is an index N such that there is an open neighborhood U_N of x and a member V_N of $\{V_n\}$, such that $f(U_N) \subset V_N \subset V$, which proves that f is continuous at x.

THEOREM 2. Let X be a space, Y be Baire, Z be a developable space with a development $\{\mathcal{P}_n\}$. If $f: X \times Y \to Z$ is quasi-continuous with respect to x, then for all $x \in X$, the set $C(f)$ is a dense G_δ subset of $\{x\} \times Y$.

Proof. Since f is quasi-continuous with respect to x, it follows from Lemma 1, that the sets $\Omega(f, \{\mathcal{P}_n\})$ are dense in $\{x\} \times Y$; by Remark 2 these sets are also open. So, $\Omega(f, \{\mathcal{P}_n\})$ is the intersection of countably many open and dense subsets of $\{x\} \times Y$. Now, we are done by Lemma 2, since every point of $\Omega(f, \{\mathcal{P}_n\})$ is a continuity point of f.

Theorem 3 below is a straightforward corollary from Theorems 1 and 2.

THEOREM 3. Let X be first countable, Y be Baire and let Z be Moore. If $f: X \times Y \to Z$ has:

a) all sections f_x quasi-continuous, and

b) all sections f_y continuous,

then $C(f)$ is a dense G_δ subset of $\{x\} \times Y$ for every $x \in X$.

The above theorem generalizes Theorem of [25] and Theorem 2 of [26], where Z is assumed to be metric.

The following two examples show that the assumptions that X is "first countable" and that Z is "Moore" in Theorem 3 are in a sense indispensable.

EXAMPLE 2 [20]. Let Y and Z denote the closed unit interval $I = [0, 1]$ and let X be the space $C_p(I, I)$ of continuous functions from I into I equipped with the pointwise topology. Then $f: X \times Y \to Z$ given by $f(x, y) = x(y)$ is separately continuous function which is discontinuous at every point of $X \times Y$.

It is worth noting that $C_p(I, I)$ is a Tychonoff space having a countable network [21] and as such it is hereditarily Lindelöf and hereditarily separable. $C_p(I, I)$ is not a Fréchet space and thus it is not first countable, see [19] and [7]. The forthcoming Example 3 shows the necessity that the range space Z is Moore.

EXAMPLE 3 [5]. Let $X = Y = [-1, 1]$ and let Z be the space of mappings from $[-1, 1]^2$ into $[-1, 1]$ equipped with the pointwise topology. Thus, $F(x, y)$ is a function of $(a, b) \in [-1, 1]^2$ given by:

$$F(x, y)(a, b) = \begin{cases} \frac{2(x-a)(y-b)}{(x-a)^2 + (y-b)^2} & \text{if this quotient is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

F is separately continuous, but not (jointly) continuous at any point. The space Z is a "large" compact Hausdorff space.

IV. Joint continuity triples and spaces of JC-type

We shall start from a few definitions. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be given classes of topological spaces. Assume $X \in \mathcal{A}$, $Y \in \mathcal{B}$ and $Z \in \mathcal{C}$. Following [27] we say that the ordered triple (X, Y, Z) is a *joint continuity triple*, denoted by JC-triple, if:

(*) for every separately continuous function

$$f: X \times Y \to Z$$

the set $C(f) \neq \emptyset$.

It follows from Namioka's theorem that (Čech-complete, σ-compact and locally compact metric) is a JC-triple.

We shall identify some other JC-triples.

The problem [30] whether (Baire, compact Hausdorff, $\mathbb{R}$) is a JC-triple, is still open.

Let $\mathcal{K}$ be a class of topological spaces and let $X \in \mathcal{K}$.

We say that X is of *JC-type* if (*) above holds for $X = Y = Z$. The family of spaces of JC-type is nonempty. In fact, by Baire's theorem *the set* $\mathbb{R}$ *of reals* is of JC-type.

By Baire-Lebesgue-Kuratowski-Montgomery theorem, every *complete metric space* is of JC-type. Really, although Namioka-type theorem does not hold here—see J. B. Brown's counterexample in [28]—every separately continuous function is of first class of Baire. Now, since the product of complete metric spaces X and Y is complete metric, hence Baire, f being pointwise discontinuous has a dense G_δ set $C(f)$.

The following Proposition 1 is a consequence of Theorem 3 in Section III.

PROPOSITION 1. *Every Baire, Moore space is of JC-type.*

Remark 3. When identifying spaces of JC-type the following results are of interest.

N. F. G. Martin [18] proved that if X is first countable, Y is Baire and Z is metric, then every separately continuous function is quasi-continuous. Now, assume $X \times Y$ is Baire, since there are metric Baire spaces X and Y such that $X \times Y$ is not Baire [12]. So, a quasi-continuous function f from the Baire space $X \times Y$ into a metric space has a dense G_δ set $C(f)$ of points of continuity. Hence, as a consequence, if X is a metric Baire space such that X^2 is Baire, then X is of JC-type. However, this result is weaker than our Proposition 1 above.

Remark 4. Due to the fact that the presence of $C(f)$ in sets of type $\{x\} \times Y$ is guaranteed, we do not have to assume that the domain $X \times Y$ of f is Baire space.

Remark 5. T. Neubrunn [22] improved the results of N. F. G. Martin, mentioned in Remark 3, by weakening the assumption on Z. Apparently, the regularity of Z suffices. Again we assume that $X \times Y$ is Baire. Now, if Z is either metric or second countable, then f being a quasi-continuous function from a Baire domain has a dense G_δ set $C(f)$ of points of continuity [22]. However, assuming the metrizability or countability of a base for Z we lose the generality of Neubrunn's assumptions.

So, a general question arises:

Assume X is a Baire space. What are "large"[1] spaces Y such that every quasi-continuous function $f : X \to Y$ has $C(f) \neq \emptyset$?

Answering this question we will show that the class of all spaces Y that have open-hereditarily countable pseudo-base and simultaneously are hereditarily Lindelöf is "too large".

In fact, we have:

EXAMPLE 4. Let X be the set $\mathbb{R}$ of reals with the Euclidean topology; let Y be the set of reals with the Sorgenfrey topology, and let $f : X \to Y$ be the identity function, i.e., $f(x) = x$, $x \in X$.

Observe that f is quasi-continuous, but the set $C(f) = \emptyset$.

Let us now turn to semitopological groups, i.e, groups in which the multiplication is separately continuous.

The following result holds:

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[1]i.e., neither metrizable, nor having a countable base

THEOREM 4. *Let X be a first countable JC-type semitopological group. Then X is paratopological.*

Proof. Let (x_0, y_0) be a continuity point of the multiplication and let (x, y) be an arbitrary but fixed element of $X \times X$. Also, assume $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$. We will show that $\lim_{n \to \infty} x_n y_n = xy$. The multiplication function will be denoted by f. Separate continuity implies:

(i) $\lim_{n \to \infty} f(x_0 x^{-1}, x_n) = \lim_{n \to \infty} (x_0 x^{-1} \cdot x_n) = x_0 x^{-1} x = x_0 e = x_0$. Also,

(ii) $\lim_{n \to \infty} f(y_n, y^{-1} y_0) = \lim_{n \to \infty} (y_n \cdot y^{-1} y_0) = y y^{-1} y_0 = e y_0 = y_0$.

Now, since (x_0, y_0) is a continuity point of the multiplication it follows:

(iii) $\forall \{(a_n, b_n)\} : (a_n, b_n) \xrightarrow[n \to \infty]{} (x_0, y_0) \implies f(a_n, b_n) \xrightarrow[n \to \infty]{} f(x_0, y_0)$. So, in particular, let $a_n = x_0 x^{-1} x_n$ and let $b_n = y_n y^{-1} y_0$.

Observe that (i) and (ii):

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (x_0 x^{-1} \cdot x_n) = x_0 \text{ and likewise}$$
$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} (y_n \cdot y^{-1} y_0) = y_0.$$

So, by (iii) we get $\lim_{n \to \infty} f(a_n, b_n) = \lim_{n \to \infty} (x_0 x^{-1} x_n \cdot y_n y^{-1} y_0) = x_0 y_0$.

(*) $\lim_{n \to \infty} (x_0 x^{-1} x_n \cdot y_n y^{-1} y_0) = x_0 y_0.$

So, $\lim_{n \to \infty} x_n y_n = \lim_{n \to \infty} e x_n y_n e = \lim_{n \to \infty} (x x^{-1}) x_n y_n (y^{-1} y) = \lim_{n \to \infty} (x e x^{-1}) x_n y_n (y^{-1} e y) = \lim_{n \to \infty} (x \cdot x^{-1} \cdot x_0 \cdot x_0^{-1}) x_n y_n (y^{-1} \cdot y_0 \cdot y_0^{-1} \cdot y) = \lim_{n \to \infty} (x \cdot x_0^{-1})(x_0 \cdot x^{-1} \cdot x_n \cdot y_n \cdot y^{-1} \cdot y_0) \cdot (y_0^{-1} \cdot y) =$

$x_n \cdot y_n \cdot y^{-1} \cdot y_0)(y_0^{-1} \cdot y) = (x \cdot x_0^{-1}) \left[\lim_{n \to \infty} (x_0 \cdot x^{-1} \cdot x_n \cdot y_n \cdot y^{-1} \cdot y_0) \right] \cdot (y_0^{-1} \cdot y) = x \cdot x_0^{-1} \cdot x_0 y_0 \cdot y_0^{-1} \cdot y = x \cdot (x_0^{-1} \cdot x_0) \cdot (y_0 \cdot y_0^{-1}) y = x e \cdot e y = xy$. □

The main result of the paper, Theorem 5 below follows now trivially from Proposition 2 and Theorem 4.

THEOREM 5. *Let X be a Baire, Moore semitopological group. Then X is paratopological.*

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