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SOME REMARKS ON CATEGORY IN TOPOLOGICAL SPACES

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ABSTRACT. We prove some results concerning the covering, additivity, and the uniform numbers for general topological spaces.

For any dense-in-itself T_1 topological space X let us put $\operatorname{cov}(X) = \inf\{|\mathcal{R}|: \mathcal{R}$ is a family of nowhere dense subsets of X covering $X\}$. Then $\operatorname{cov}(X)$ is always an infinite cardinal that does not exceed $|X|$, the cardinality of X. If $\operatorname{cov}(X)$ is an uncountable cardinal, then X is second category, and if $\operatorname{cov}(U)$ is uncountable for every nonempty open subset of X, then X is a Baire space.

Let us put $\operatorname{cov}^2(X) = \inf\{\operatorname{cov}(F): F$ is a closed, nonempty, dense-in-itself subspace of $X\}$. If $\operatorname{cov}^2(X)$ is an uncountable cardinal, then X is usually called a totally nonmeager space [1] (sometimes it is also called a totally inexhaustible space [8]).

In [8], the behavior of totally nonmeager spaces under feebly continuous and feebly open mappings have been considered (see also [2]). Let us recall [4] that a mapping f of a space X onto a space Y is feebly continuous if $\operatorname{int} f^{-1}(V) \neq \emptyset$ whenever V is nonempty and open in Y, and f is feebly open if $\operatorname{int} f(U) \neq \emptyset$. whenever U is nonempty and open in X. T. Neubrunn [9] conjectured that

(*) If f is a one-to-one feebly continuous and feebly open mapping of a regular space X onto a totally nonmeager space Y, then X is a Baire space.

We shall show that this conjecture is true even without appealing to any additional separation axioms. For this purpose we prove the following general result.

THEOREM 1. *Let f be a one-to-one feebly continuous and feebly open mapping of a space X onto a space Y. If U is a nonempty open subset of X, then* $\operatorname{cov}^2(Y) \leq \operatorname{cov}(U)$.

PROOF. Suppose that U is a nonempty open subset of X, $\mathcal{R}$ is a family of nowhere dense subsets of U that covers U, and $|\mathcal{R}| = \operatorname{cov}(U)$. Let us put $W = \operatorname{int}\operatorname{cl}\operatorname{int} f(X - \operatorname{cl} U)$. Then W is an open subset of Y that is disjoint with $\operatorname{int} f(U)$, f being one-to-one. We shall show that $\operatorname{cov}(F) \leq \operatorname{cov}(U)$, where $F = Y - W$. The closed set F can be decomposed into two disjoint parts: $F_1 = \operatorname{int} f(U)$ and $F_2 = F - \operatorname{int} f(U)$. Part F_2 is a closed subset of the closed set F. Let us check that it is also nowhere dense in F. So let V be an open subset of Y such that $F_2 \cap V \neq \emptyset$. Hence $V - W \neq \emptyset$ and, because V is open $G = V - \operatorname{cl}\operatorname{int} f(X - \operatorname{cl} U) \neq \emptyset$. Because f is feebly continuous, $\operatorname{int} f^{-1}(G) \neq 0$. This set has to be disjoint with $X - \operatorname{cl} U$; otherwise $G \cap f(X - \operatorname{cl} U) \neq \emptyset$ (f being feebly open), which is impossible. Hence $\operatorname{int} f^{-1}(G) \subset \operatorname{cl} U$, and therefore $\operatorname{int} f^{-1}(G) \cap U \neq \emptyset$. Because f is feebly open,

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$G \cap \operatorname{int} f(U) \neq \emptyset$. Since $G \subset V$, $V \cap \operatorname{int} f(U) \neq \emptyset$ which shows that F_2 is nowhere dense in F.

To prove our theorem now, it is enough to show that $\operatorname{int} f(U)$ can be covered by $\operatorname{cov}(U)$ or a less nowhere dense subset of F. We will have this if we show that for any $E \in \mathcal{R}$, $F_E = \operatorname{int} f(U) \cap f(E)$ is a nowhere dense subset of F. To see this we argue in the following way. The image of a nowhere dense subset of X under a one-to-one feebly open and feebly continuous mapping is again a nowhere dense subset of Y. Nowhere dense subsets of an open set of a space are exactly those sets which are subsets of the open set and nowhere dense in the space. Therefore each F_E is a nowhere dense subset of $\operatorname{int} f(U)$, being the intersection of a nowhere dense set in the space Y with an open subset of the space Y. Because $\operatorname{int} f(U)$ is contained in F, those sets are also nowhere dense in F. □

As we know, $\operatorname{cov}^2(Y) > \omega$ means the same as Y being totally nonmeager, and $\operatorname{cov}(U) > \omega$ for each nonempty open subset of X means the same as X being a Baire space, so (∗) follows immediately from Theorem 1.

Our next topic is to consider the cardinal $\operatorname{add}(X)$, the additivity of the category for the space X. In order for $\operatorname{add}(X)$ to be defined we restrict ourselves to the case in which $\operatorname{cov}(X)$ is defined and $\operatorname{cov}(X) > \omega$. In such a case we put

$$\operatorname{add}(X) = \inf \Big\{ |\mathcal{R}| : \mathcal{R} \text{ is a family of nowhere dense subsets of } X \text{ and } \bigcup \mathcal{R} \text{ is not first category in } X \Big\}.$$

There is an obvious inequality: $\operatorname{add}(X) \leq \operatorname{cov}(X)$. Both cardinals $\operatorname{add}(X)$ and $\operatorname{cov}(X)$ have been intensively studied for X being a metric separable space (see, for example, [6, 7]). Many interesting and deep results have been obtained. One of the most celebrated results in this topic is the following inequality:

(∗∗) If κ is a cardinal such that $P(\kappa)$ holds and X is a separable, metric second category space, then $\operatorname{add}(X) > \kappa$.

Here $P(\kappa)$ denotes the following combinatorial statement:

$P(\kappa)$: If S is a family of infinite subsets of ω such that $|S| \leq \kappa$, and any finite subfamily of S has an infinite intersection, then there is an infinite subset A of ω such that $A - s$ is finite for each $s \in S$.

The result (∗∗) has been shown by D. Martin and R. Solovay [5]. We shall present a theorem that generalizes (∗∗). Our generalization goes in two directions. First, it concerns a much wider class of spaces. Second, it uses a weaker assumption than $P(\kappa)$—namely, we shall use the following statement:

$P(\kappa)\downarrow$. If $\{T_\alpha : \alpha < \lambda\}$ is a family of infinite subsets of ω such that $\lambda \leq \kappa$ and $\alpha < \beta < \lambda$ implies $T_\beta - T_\alpha$ is finite, then there is an infinite subset A of ω such that $A - T_\alpha$ is finite for each $\alpha < \lambda$.

Clearly, $P(\kappa)\downarrow$ follows from $P(\kappa)$. It is known [3] that $P(\omega_1)$ follows from $P(\omega_1)\downarrow$. However it is unknown whether $P(\kappa)$ is equivalent to $P(\kappa)\downarrow$, in general.

THEOREM 2. *If $P(\kappa)\downarrow$ holds and X is a second category space containing a dense countable subspace of points with countable character, then $\operatorname{add}(X) > \kappa$.*

To prove this theorem we will need two lemmas. The first of these is already known; it goes back to F. Rothberger [10] though the explicit proof can be found in [3].

LEMMA 1. *Let $\mathcal{F}$ be a family of functions from ω into ω such that $|\mathcal{F}| \leq \kappa$. If $P(\kappa)\downarrow$ holds, then there is a function g from ω into ω such that the set $\{n \in \omega: g(n) \leq f(n)\}$ is finite for each $f \in \mathcal{F}$.* □

LEMMA 2. *Let X be a dense-in-itself T_1 space with a countable π-base, and let $\{D_\alpha: \alpha < \lambda\}$ be a family of countable dense subsets of X such that $D_\beta - D_\alpha$ is finite whenever $\alpha < \beta < \lambda$. If $\lambda \leq \kappa$ and $P(\kappa)\downarrow$ holds, then there is dense subset D of X such that $D - D_\alpha$ is finite for each $\alpha < \lambda$.*

PROOF. Let $\mathcal{P}$ be a countable π-base in X consisting of nonempty open sets. For each $U \in \mathcal{P}$ let us consider the family $\{U \cap D_0 \cap D_\alpha: \alpha < \lambda\}$. Each such family consists of infinite subsets of the countable set $U \cap D_0$. This holds because X is T_1 and the sets D_α are dense in X. We also have $U \cap D_0 \cap D_\beta - U \cap D_0 \cap D_\alpha$ is finite, whenever $\alpha < \beta < \lambda$. So we may apply $P(\kappa)\downarrow$, and we get that for any $U \in \mathcal{P}$ there is an infinite subset D_U of $U \cap D_0$ such that $D_U - D_\alpha$ is finite for each $\alpha < \lambda$. Enumerate D_U by $\{d(U,n): n \in \omega\}$. Now for each $\alpha < \lambda$ define a function $f_\alpha: \mathcal{P} \to \omega$, setting $f_\alpha(U) = \min\{n: d(U,m) \in D_\alpha \text{ for each } m \geq n\}$. Since $D_U - D_\alpha$ is finite for each $U \in \mathcal{P}$, the function f_α is well defined. We may apply our Lemma 1 to the family $\{f_\alpha: \alpha < \lambda\}$, and we get the existence of a function g from $\mathcal{P}$ into ω such that the set $\{U \in \mathcal{P}: g(U) \leq f_\alpha(U)\}$ is finite for each $\alpha < \lambda$. Let us put $D = \{d(U,n): U \in \mathcal{P} \text{ and } g(U) \leq n\}$. Then D intersects each D_U on an infinite set and therefore D intersects each $U \in \mathcal{P}$ on an infinite set. Since $\mathcal{P}$ is a π-base in X, D is dense in X. It remains to be shown that $D - D_\alpha$ is finite for each $\alpha < \lambda$. So fix α, $\alpha < \lambda$. Let us observe that if $d(U,n) \in D - D_\alpha$, then $g(U) \leq f_\alpha(U)$ and $d(U,n) \in D_U - D_\alpha$. Hence $D - D_\alpha \subset \bigcup\{D_U - D_\alpha: U \in \mathcal{P} \text{ and } g(U) \leq f_\alpha(U)\}$. Because the sets $D_U - D_\alpha$ are finite and the set $\{U \in \mathcal{P}: g(U) \leq f_\alpha(U)\}$ is finite as well, $D - D_\alpha$ is a finite set.

PROOF OF THEOREM 2. Assume, to the contrary, that there exists in the space X a family $\mathcal{R}$ consisting of nowhere dense subsets of X such that $\bigcup \mathcal{R}$ is second category in X and yet $|\mathcal{R}| \leq \kappa$. Without loss of generality we may assume that X is dense-in-itself (otherwise apply the arguments below to the set $\operatorname{int}\operatorname{cl}\bigcup\mathcal{R}$). Let D be a countable dense subset of X consisting of points with countable character. For each $d \in D$ enumerate a countable base around d by $\{U(d,n): n \in \omega\}$. We may also assume that $U(d,n) \subset U(d,m)$, whenever $m \leq n$. Of course, the family $\{U(d,n): d \in D \text{ and } n \in \omega\}$ forms a countable π-base in X. Knowing this we will be able to find a dense subset C contained in D such that $C \cap \operatorname{cl} F$ is finite for each $F \in \mathcal{R}$. For this purpose enumerate the family $\mathcal{R}$, say $\mathcal{R} = \{F_\alpha: \alpha < \lambda\}$. Since $|\mathcal{R}| \leq \kappa$, $\lambda \leq \kappa$. And now, we shall inductively define a family $\{D_\alpha: \alpha < \lambda\}$ satisfying the following conditions:

(i) $D_\alpha \subset D$, D_α is dense in X, and $D_\alpha \cap \operatorname{cl} F_\alpha = \emptyset$ for each $\alpha < \lambda$;

(ii) if $\alpha < \beta < \lambda$, then $D_\beta - D_\alpha$ is finite.

Put $D_0 = D - \operatorname{cl} F_0$. Assume $\gamma < \lambda$ and $\{D_\alpha: \alpha < \gamma\}$ has already been defined. If γ is a nonlimit ordinal, say $\gamma = \beta + 1$, then it is enough to put $D_\gamma - D_\beta - \operatorname{cl} F_\gamma$. If γ is a limit ordinal, then let $\{\alpha_\xi\ \xi < \operatorname{cf}(\gamma)\}$ be a strongly increasing sequence of ordinals less than γ. Because $\operatorname{cf}(\gamma) < \lambda \leq \kappa$ and $\operatorname{cf}(\gamma)$ is a cardinal, we may apply our Lemma 2 to the family $\{D_{\alpha_\xi}: \xi < \operatorname{cf}(\gamma)\}$. Hence we get a dense subset $\tilde{D}_\gamma$ of X such that $\tilde{D}_\gamma - D_{\alpha_\xi}$ is finite for each $\xi < \operatorname{cf}(\gamma)$. We shall show that $\tilde{D}_\gamma - D_\alpha$ is finite for each $\alpha < \gamma$. So fix α, $\alpha < \gamma$. There is ξ, $\xi < \operatorname{cf}(\gamma)$, such that $\alpha < \alpha_\xi$.

Hence

$$\tilde{D}_\gamma - D_\alpha \subset \tilde{D}_\gamma - (D_\alpha \cap D_{\alpha_\xi}) \subset (\tilde{D}_\gamma - D_{\alpha_\xi}) \cup (D_{\alpha_\xi} - D_\alpha)$$

which is a finite set. Finally, if we put $D_\gamma = (\tilde{D}_\gamma \cap D) - \mathrm{cl}\, F_\gamma$ we get the desired set to complete the induction step.

Now, if we have a family $\{D_\alpha : \alpha < \lambda\}$ satisfying conditons (i) and (ii), we shall again apply our Lemma 2 to this family (since $\lambda \le \kappa$), and we get a dense subset C of X such that $C - D_\alpha$ is finite for each $\alpha < \lambda$. We may assume that C is a subset of D (throwing out finitely many points if needed); the properties $C - D_\alpha$ is finite and $D_\alpha \cap \mathrm{cl}\, F_\alpha = \emptyset$ give us that $C \cap \mathrm{cl}\, F_\alpha$ is finite so $C \cap \mathrm{cl}\, F$ is finite for each $F \in \mathcal{R}$.

Enumerate the points of C by $\{c_n : n \in \omega\}$ and consider the families $\mathcal{R}_n = \{F \in \mathcal{R} : C \cap \mathrm{cl}\, F \subset \{c_0, \dots, c_n\}\}$. Each member of $\mathcal{R}$ falls into one of the families $\mathcal{R}_n$. Hence one of them, say $\mathcal{R}_m$, has the union of its members being second category. Let us throw out $m+1$ first points from the set C. Then we obtain a dense countable set B such that $B \cap \mathrm{cl}\, F = \emptyset$ for each $F \in \mathcal{R}_m$. Now for each $F \in \mathcal{R}_m$ define a function $f_F : B \to \omega$ setting $f_F(b) = \min\{n : U(b,n) \cap \mathrm{cl}\, F = \emptyset\}$ (recall that B is a subset of D and for each $d \in D$ we have assigned a local base, $\{U(d,n) : n \in \omega\}$). Since $b \notin \mathrm{cl}\, F$ for each $F \in \mathcal{R}_m$, the function f_F is well defined. We may apply our Lemma 1 to the family $\{f_F : F \in \mathcal{R}_m\}$, and we get the existence of a function $g : B \to \omega$ such that the set $\{b \in B : g(b) \le f_F(b)\}$ is finite for each $F \in \mathcal{R}_m$. For each finite subset s of the set B let us put $E_s = X - \bigcup\{U(b, g(b)) : b \in B - s\}$. The number of the sets E_s is countable, since B is a countable set. So we will come to a contradiction if we show that $\bigcup \mathcal{R}_m \subset \bigcup\{E_s : S$ is a finite subset of $B\}$ and that each set E_s is nowhere dense in X.

To see that $\bigcup \mathcal{R}_m \subset \{E_s : s$ is a finite subset of $B\}$, let us observe that $F \subset E_s$, whenever $s = \{b \in B : g(b) \le f_F(b)\}$.

To see that E_s is nowhere dense, take an arbitrary nonempty open subset V of X. The $V \cap B$ is infinite and therefore there is a point b_0 belonging to $V \cap B - s$. Consequently,

$$b_0 \in V \cap U(b_0, g(b_0)) \subset V \cap \bigcup\{U(b, g(b)) : b \in B - s\}. \quad \square$$

Our final topic is to consider the cardinal $u(X)$. It is defined (if possible) in the following way:

$$u(X) = \inf\{|Y| : Y \text{ is a second category subset of } X\}.$$

There are obvious inequalities between $\mathrm{add}(X)$ and both cardinals $\mathrm{cov}(X)$ and $u(X)$ in the case in which $\mathrm{cov}(X)$ is defined and $\mathrm{cov}(X) > \omega$, namely $\mathrm{add}(X) \le u(X)$ and $\mathrm{add}(X) \le \mathrm{cov}(X)$. Within ZFC one cannot prove the equalities $\mathrm{add}(X) = u(X)$ or $u(X) = \mathrm{cov}(X)$, even if X is the space of reals [6].

$M(\kappa)$. If $\mathcal{F}$ is a family of functions from ω into ω such that $|\mathcal{F}| \le \kappa$, then there is a function g from ω into ω such that the set $\{n \in \omega : g(n) = f(n)\}$ is finite for each $f \in \mathcal{F}$.

THEOREM 3. *If $M(\kappa)$ holds and X is a dense-in-itself second category Hausdorff space with a countable π-base, then $u(X) > \kappa$.*

PROOF. Assume, to the contrary, that the space X contains a second category subset Y such that $|Y| \le \kappa$. Let $\mathcal{P}$ be a countable π-base in X. Because X is a dense-in-itself Hausdorff space, each nonempty open subset of X contains infinitely many disjoint nonempty open subsets. For any $u \in \mathcal{P}$, $u \neq \emptyset$, let $\{u(n): n \in \omega\}$ be a disjoint family of nonempty open subsets of the set u. Now, for any $y \in Y$ define a function $f_y: \mathcal{P} - \{\emptyset\} \to \omega$ setting $f_y(u) = n$ if $y \in u(n)$ and $f_y(u) = 0$ if $y \notin \bigcup\{u(n): n \in \omega\}$. Because $|Y| \leq \kappa$ and $M(\kappa)$ holds, there is a function g from $\mathcal{P} - \{\emptyset\}$ into ω such that the set $\{u \in \mathcal{P} - \{\emptyset\}: g(u) = f_y(u)\}$ is finite for each $y \in Y$. For each finite subset s of the set $\mathcal{P}$ let us put $E_s = X - \bigcup\{u(g(u)): u \in \mathcal{P} - s\}$. The number of the sets E_s is countable. So, we will come to a contradiction if we show that $Y \subset \bigcup\{E_s: s \text{ is a finite subset of } \mathcal{P}\}$ and that each set E_s is nowhere dense. To see the first observe that if $y \in Y$, then $y \in E_s$, where $s = \{u \in \mathcal{P} - \{\emptyset\}: g(u) = f_y(u)\}$. To see that each E_s is nowhere dense, fix s and take an arbitrary nonempty open subset V of X. Because X is dense in itself, the set of all members of $\mathcal{P}$ contained in V is infinite, and therefore there is one in $\mathcal{P} - (s \cup \{\emptyset\})$. Hence $V \cap \bigcup\{u(g(u)): u \in \mathcal{P} - s\} \neq \emptyset$.

In [**6**, Theorem 1.3] some analogous results have been obtained for metric separable spaces.

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