1. Blumberg theorem.

In 1922 H.Blumberg [A1] proved that if $X = Y = \mathbf{R}$, the reals then :

(*) for every $f:X\to Y$, there exits $D \subset X$, D dense in X such that $f|D$ is continuous.

Even for functions $f: \mathbf{R}\to\mathbf{R}$, the set D in (*) cannot be made to have cardinality c, see [A2]. (S.Baldwin,[2] in [A5] showed recently that it is consistent with the axioms of set theory that the set D in (*) can always be chosen to be uncountably dense); the set D cannot be necessarily chosen so that for 1-1 functions, $f|D$ is a homemorphism, see C.Goffman in [9] of [A5]; further, the set D cannot necessarily be chosen so as to make $f|D$ differentiable or monotonic, see J.Cedar [16] in [A5]. Despite the above, J.B.Brown [6] in [A5] proved that for every $f:\mathbf{R}\to\mathbf{R}$ there exists a set $N \subset \mathbf{R}$, N c-dense in $\mathbf{R}$ such that $f|N$ is pointwise discontinuous (rel N).

2. Blumberg spaces.

Let $Y = \mathbf{R}$. Call a space X **Blumberg** if (*) holds for X. J.C. Bradford and C. Goffman [A3] proved that a metric space X is Blumberg if and only if X is Baire, i.e., a space in which open, nonempty subsets are of second category. The key lemma in their proof is Banach Category theorem. H.E. White Jr. [A4] extended Bradford-Goffman result to topological spaces X which have σ-disjoint pseudobases. he also showed that the reals $\mathbf{R}$ with the density topology is a Baire space not being Blumberg W.A.R. Weiss in [9] of [A5] gave an example of a compact Hausdorff space which is not Blumberg. Z.Piotrowski and A.Szymanski [A6] showed that if X is a space for which (*) holds with $Y = \mathbf{R}$. then this implies that (*) holds for every 2^{nd} countable space Y. The following characterization is of interest, see [A6]. A space is Blumberg if and only if for every countable cover $\mathcal{P}$ of X, there exists a dense subset D of X such that $P \cap D$ is open in D, for every $P \in \mathcal{P}$, see also [A7] for more characterizations.

General references to this area are [A5] and [9] of [A5].

A simple example of the identity function f from the reals with the Euclidean topology into the reals with the discrete topology exhibits the necessarily of some type of restrictious placed upon the range space. Spaces Y for which (*) holds, X being $\mathbf{R}$, were studied in [A8].

3. The dynamics of Blumberg spaces.

It is easy to see that a dense, or a closed subspace of a Blumberg space need not be Blumberg. The Stone-Cech compactification of a dense subspace of a completely regular Blumberg space is a Blumberg space, a result due to R.Levy and R.H. McDowell,[4] in [A6].

The Cartesian product of Blumberg spaces need not to be a Blumberg space, since there is a metric Baire space (hence Blumberg) whose square is not Baire. On the other hand, S. Todorcevic [A9] showed that there is a first countable space X, which is not Blumberg, whereas $X \times X$ is a Blumberg space. It follows from the above theorem that the image of a Blumberg space under an open and continuous function need to be Blumberg.

Consider the union X of the graph Z of the Riemann function, restricted to the rationals and the copy Y of the rationals in the x-axis. The natural projection of X onto Y (which is constant on Y) shows that even perfect, continuous functions do not necessarily preserve Blumberg spaces. In contrast, [A6] Blumberg spaces are preserved in preimages under irreducible surjections.

4. Varia.

M. Valdivia [A10] showed that (*) holds for *linear* transformations, where X and Y are metrizable linear spaces, and X is of the second category. L.Drewnowski [A11] proved that "dense subset" in Valdivia's theorem cannot be replaced with "dense linear subspace"

Also Blumberg sets - dense sets D, appearing in (*) have been studied in connection with the characterizations of some almost continuous functions, e.g., quasicontinuous see [A12], section 6.

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