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SOME REMARKS ON ALMOST CONTINUOUS FUNCTIONS *)

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I. Introduction

Nearly continuous functions (known also as *almost continuous*, or *densely approaching* functions) arise naturally when we study Open Mapping and the Closed Graph Theorem [3] (in fact, this "somewhat continuity" goes back to S. Banach, compare [5] p. 214) or the Blumberg Theorem [1].

A function $f: X \to Y$ is said to be *nearly continuous* at $x_0 \in X$ if for every open $V \ni f(x_0)$ the set $\mathrm{Cl} f^{-1}(V)$ contains a neighborhood of x_0. We say that f is *nearly continuous* if it is nearly continuous at every $x \in X$.

We shall consider here a relationship between separate and joint near continuity (on product spaces).

Generally, neither separate near continuity does imply joint near continuity, nor vice versa [6].

T. Neubrunn ([6]. Example 1, p. 308) proved, however, that there is a nearly continuous function $f: [-1, 1]^2 \to R$ which is not separately nearly continuous at (0, 0); i.e. none of the sections f_x and f_y is nearly continuous at 0. Next, he proved the following.

Proposition 1. ([6] Thm. 1, p. 308—310). *Let X, Y be separable metric spaces which are dense-in-themselves. Then there exist a real function $f: X \times Y \to R$ such that f is nearly continuous on $X \times Y$ and there is countable, dense set $C \subset X \times Y$ such that for each $(x_0, y_0) \in C$, the sections f_{x_0} and f_{y_0} are not nearly continuous.*

In the following Example 1, which illustrates clearly Neubrunn's result, we explicitely construct a function for a special, important case. The existence of such a function is proved in Proposition 1 (of course for a wider class of spaces).

Example 1. Let $I = [0, 1]$ and let R be the set of reals. Put $D_n = \{(x, y):$

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$x = \frac{k}{2^n}$, $y = \frac{p}{2^n}$, where k and p are all odd numbers between 0 and $2^n\Big\}$. Let $D = \bigcup_{n=1}^{\infty} D_n$. It is easy to see that $\mathrm{Cl}\, D = I^2$. Now, let us define $f\colon I^2 \to R$ by: $f(x, y) = 1$, for $(x, y) \in D$ and $f(x, y) = 0$ if $(x, y) \notin D$. The function f is nearly continuous and for $(x_0, y_0) \in D$ the sections f_{x_0} and f_{y_0} are not; every such section has finitely many "points of the jump" of f.

Our approach in Example 1, of using the characteristic function of a special "thin" subset of the product, was based on the following observations:

1. The characteristic function of a dense co-dense subset of a space *is* nearly continuous.
2. The characteristic function of a nowhere dense subset is not nearly continuous at points of this set.

So, combining 1. and 2., if D is a dense subset of the product $X \times Y$ such that its intersection with any vertical and horizontal line is nowhere dense in these lines, then its complement is again dense, so that the characteristic function of such a set D is nearly continuous but the sections f_x and f_y are not nearly continuous for every x belonging to the projection of the set D on X and for every y belonging to the projection of the set D on Y.

In particular, when D has at most *one point* in common with vertical and horizontal lines, then the sections are not nearly continuous on at most one point. Clearly, such functions can be viewed as "minimal" with respect to this property.

In the forthcoming section we shall construct special dense subsets with the properties we have just discussed.

II. Special subsets of product spaces

Following [8] we have the following two definitions. Assume $X = \prod_{\alpha < \varkappa} X_\alpha$ and let $D \subset X$.

We say that D is *thin* if whenever $x, y \in D$ and $x \neq y$, then

$$|\{\alpha < \varkappa\colon x_\alpha \neq y_\alpha\}| > 1.$$

A set D is said to be *very thin* if for every $\alpha < \varkappa$ and $p \in X_\alpha$,

$$|\{x \in D\colon x_\alpha = p\}| \leqslant 1.$$

Clearly, every very thin set is thin. In the product of two spaces, these two notions of thinness coincide.

Geometrically, in the case $\varkappa = 3$, no two points of a thin set in I^3 can be on a *line* parallel to either of the axes: i.e. if D is thin and $x \in D \subset I^3$, $x = (x_0, y_0, z_0)$, then x is the only point in

$$[(\{x_0\} \times \{y_0\} \times Z) \cup (\{x_0\} \times Y \times \{z_0\}) \cup (X \times \{y_0\} \times \{z_0\})] \cap D.$$

Similarly, in the case of a very thin set D, no two points can lie on a *plane* parallel to one of the three main planes, i.e. if $x \in D \subset I^3$, then x is the only point in

$$[(\{x_0\} \times Y \times Z) \cup (X \times \{y_0\} \times Z) \cup (X \times Y \times \{y_0\})] \cap D.$$

Proposition 2. (see [8]) *Let X be the product of 2^ω many dense-in-themselves, separable spaces X_α with countable dense subsets $D_\alpha = \{x(\alpha, n): n \in \omega\}$, respectively. Then there is a countable, dense and thin subset of X.*

The assumption that all spaces X_α are dense-in-themselves is essential. In fact, let $X = [0, 1] \cup \{2\}$ and $Y = [0, 1]$. Consider $X \times Y$ as a subspace of the plane. It is clear that every countable dense subset of $X \times Y$ has infinitely many points in the fiber $\{2\} \times Y$.

Following J. C. Oxtoby [7] we say that a family $\mathscr{B}$ of non-empty open sets in a space is a *pseudo-base* if every non-empty open set contains at least one member of $\mathscr{B}$. It is clear that every base is a pseudo-base and that every space having a countable pseudo-base is separable.

The Stone–Čech compactification βN of naturals is an example of a space having a countable dense set of isolated points (hence, it has a countable pseudo-base) but no countable base, cf. [7] p. 159.

The property of possessing a countable pseudo-base is intermediate between that of having a countable base and that of being a separable space. For metrizable spaces all three properties are equivalent, see [7].

Proposition 3. (see [8]) *Let X be the product of 2^ω many dense-in-themselves spaces with countable pseudo-base $\mathscr{B}_\alpha$ respectively. Then there is a countable dense and very thin set D in X.*

III. Near continuity on products

Let $X = \prod_{\alpha < \kappa} X_\alpha$ and let $x = (x_\alpha) \in X$.

The set $W_\alpha(x) = \{y \in X: x_\alpha = y_\alpha\}$ will be called the *α-wall of x in X*.

The set $L_\alpha(x) = \{y \in X: (\beta \neq \alpha) \Rightarrow (x_\beta = y_\beta)\}$ will be called the *α-line through x in X*.

Observe that a set $A \subset X$ is thin (resp. very thin) if for every $x \in X$ and every $\beta < \kappa$, the β-line through x in X (resp. α-wall of x in X) has at most one element of A.

Consequently we have the following two notions of "separate" near continuity on arbitrary products.

A function $f\colon \prod_{\alpha<\varkappa} X_\alpha \to Y$ is called *L-nearly continuous* (resp. *W-nearly continuous*) if $\forall x \in \prod_{\alpha<\varkappa} X_\alpha \forall \underset{\text{open}}{V} \ni f(x) \exists \beta < \varkappa$: $\mathrm{Cl} f^{-1}(V) \cap L_\beta(x)$ (resp. $\mathrm{Cl} f^{-1}(V) \cap W_\beta(x)$) is a neighbourhood of x in $L_\beta(x)$ (resp. in $W_\beta(x)$).

The following two statements easily follow from Propositions 2 and 3. They both strongly generalize T. Neubrunn's result, Proposition 1.

Proposition 4. *Let X be the product of 2^ω many separable (resp. having countable pseudo-bases) dense-in-themselves spaces and let Y be a Hausdorff space containing at least two points.*

Then there is a nearly continuous function f from X into Y and there is a countable dense set $D \subset X$ such that for every $x \in D$, f is not L-nearly continuous (resp. W-nearly continuous).

Remark. As it is observed in [8], Propositions 2 and 3 hold for any $\varkappa > \omega$, where separability and countable pseudo-basis are replaced by density $\varkappa$ and π-weight $\varkappa$ respectively. Hence Proposition 4 can be even further generalized.

IV. A dual problem

The following Lemma 5 will be used in the sequel.

Lemma 5. *Let X be arbitrary, Y be regular and let $f\colon X \to Y$ be quasicontinuous and nearly continuous at x_0. Then f is continuous at x_0.*

Proof: Suppose f is not continuous at x_0. Then there exists V open, $V \ni f(x_0)$ such that for any open U containing x_0 there is $x \in U$ such that $f(x) \notin V$. Take W open such that $f(x_0) \in W \subset \mathrm{Cl}\, W \subset V$. Using near continuity at x_0 we have an open neighbourhood U_1 of x_0 and a dense set $H \subset U_1$ such that $f(H) \subset W$. Take $x_1 \in U_1$ such that $f(x_1) \notin V$. Then by quasicontinuity of f at x_1 there exists a nonempty open set $G \subset U_1$ such that $f(G) \subset Y \setminus \mathrm{Cl}\, W$, which is a contradiction with $f(H) \subset W$.

T. Neubrunn [6] in Ex. 2 p. 310 showed that there is a separately nearly continuous function $f\colon I^2 \to R$ which is not nearly continuous at $(0, 0)$. However, the following general result is true.

Proposition 6. *Let X be Baire, Y be a space such that any point $y \in Y$ possesses a neighbourhood satisfying the second countability axiom and let Z be regular, or let X, Y and Z be metrizable and let $f\colon X \times Y \to Z$ be any separately continuous function which is not continuous on a set $D \subset X \times Y$. Then any point of D is a point at which f is not nearly continuous.*

Proof: Clearly, as separately continuous, such f is separately nearly continuous. So, we only need to show that f is not nearly continuous on D.

Case I. X is Baire. Y-every point $y \in Y$ possesses a neighbourhood satisfying the second countability axiom. Z is regular. Suppose, to the contrary, that f is nearly continuous on D. Then as separately continuous, it is, by [6] Thm. 2 quasi-continuous ($\equiv$ the inverse image of every open set lies between an open set and its closure) on D. And now, by Lemma 5 we get that f is continuous on D, which is is a contradition.

Case II. Let X, Y and Z be metric. Again, we suppose that f is nearly continuous. Then as separately continuous, it has, by [4] Thm. 4, p. 148, the Baire property. And now by [9] Thm. 4, p. 152, f is continuous, a contradiction.

Remark 7. Obviously it is sufficient to assume in Proposition 6, Case I, that f_x, f_y are quasi-continuous. Moreover, it is sufficient to suppose (see [6] Thm. 2) that f_x's are quasi-continuous with the exception of a set of the first category.

It is not clear [6] whether a dual question to that answered by Thm. 1 of [6] has the positive answer for some classes of spaces X, Y and Z. Here is this dual problem:

Let X, Y and Z be "nice" spaces. Does there exist a function $f: X \times Y \to Z$ and a dense set $D \subset X \times Y$ such that f is separately nearly continuous or rather, in view of Proposition 6, separately continuous, and which is not nearly continuous at every point of D?

The first and already completely satisfactory answer to this question in case $X = Y = Z = I$ came in 1910(!) by W. H. Young and G. C. Young [10].

They showed an example of a function which is continuous along every straight line (in I^2) but which has *uncountably* many points of discontinuity in *every* rectangle contained in I^2. J. C. Breckenridge and T. Nishiura ([2] Thm. 5.2, p. 201) showed that:

If A and B are sets of first category in metric spaces X and Y, respectively, if H is an F_σ-subset of $X \times Y$ with $H \subset A \times B$, then there is a separately continuous function $f: X \times Y \to R$ (= the real line) such that H is the set of points of discontinuity. The following Proposition 8 is derived from the above result and Proposition 6; it gives an andwer to our dual problem.

Proposition 8. *Let X and Y be metric, separable dense-in-themselves spaces. Then there is a real-valued, separately continuous function $f: X \times Y \to R$ and there is a dense subset $D \subset X \times Y$ such that f is not nearly continuous at points of D.*

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НЕКОТОРЫЕ ЗАМЕЧАНИЯ О ПОЧТИ НЕПРЕРЫВНЫХ ФУНКЦИЯХ

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Резюме

В работе даются некоторые применения теорем Хьюита – Марчевского – Пондичеры к почти непрерывным функциям. Изучается обобщенная непрерывность по каждой переменной отдельно на произведение большого количества пространств.