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A solution of a problem by M. Henriksen and R.G. Woods

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Abstract

Henriksen and Woods [Topology Appl. 97 (1999) 175–205, Problem (C), p. 203] asked whether there are Tychonoff spaces X and Y with $X \times Y$ being Baire such that:

(a) Every separately continuous function $f : X \times Y \to \mathbb{R}$ has a dense (in fact: G_δ) set $C(f)$ of points of continuity;
(b) There exists a separately continuous function $g : X \times Y \to \mathbb{R}$ for which $C(g)$ fails to contain either $A \times Y$ or $X \times B$ for any dense G_δ set $A \subset X$ or dense G_δ set $B \subset Y$.

We will answer this question by showing the spaces X and Y can even be complete metric and condition (b) can be strengthened to the following: There exists a separately continuous function $g : X \times Y \to \mathbb{R}$ so that if $C(g)$ contains either $A \times Y$ or $X \times B$, then both A and B are empty.

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1. Big quadrant

Let $X = Y = \bigoplus_{\alpha \in [0,1]} [0,1]_\alpha$ be the topological sum of spaces $[0,1]_\alpha$, $\alpha \in [0,1]$ metrized with the metric d, defined as follows:

$$d(x, y) = \begin{cases} |x - y|, & \text{if both } x \text{ and } y \text{ belongs to the same } [0,1]_\alpha,\ 0 \leqslant \alpha \leqslant 1, \\ 1, & \text{otherwise.} \end{cases}$$

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(X, d) is obviously a complete metric space. Moreover, we can think of X as an ordered set: within each $[0, 1]_\alpha$ we have the usual ordering and if $x \in [0, 1]_\alpha$, $y \in [0, 1]_\beta$ with $\alpha < \beta$ then $x < y$. Similarly for Y.

Now consider $X \times Y = (\bigoplus_{\alpha\in[0,1]}[0, 1]_\alpha) \times (\bigoplus_{\alpha\in[0,1]}[0, 1]_\alpha)$; think of it as a matrix consisting of $c \times c$ many squares $S_{r,s} = [0, 1]_r \times [0, 1]_s$, $0 \leqslant r \leqslant 1$, and $0 \leqslant s \leqslant 1$, having c many "rows" and c many "columns". Each square $S_{r,s}$ has its own local coordinate system.

2. Condition (a)

By the Kuratowski–Montgomery theorem [1, Theorem 3.3, p. 299], any separately continuous function f is class 1 of Baire as a real-valued separately continuous function defined on a product of two metric spaces. As such, f is pointwise discontinuous; that is, the set $D(f)$ of discontinuity points is of first category. Thus $C(f)$ is residual. As a product of two complete metric spaces, the product $X \times Y$ is Baire, so $C(f)$, in fact, *is* a dense G_δ subset of $X \times Y$, since the range of f, the reals, is a metric space. Therefore condition (a) mentioned in Abstract is met.

3. Condition (b)

We shall now prove that there is a separately continuous function $g : X \times Y \to \mathbb{R}$ for which $C(g)$ contains neither $A \times Y$ nor $X \times B$ for any dense G_δ set $A \subset X$ or dense G_δ set $B \subset Y$.

In what follows we shall construct a set $D \subset X \times Y$ of the form $D = \{D_{r,s}: (r, s) \in [0, 1] \times [0, 1]\}$ where for a fixed pair (r, s), $D_{r,s}$ is a point from the square $S_{r,s}$ lying on its main diagonal, i.e., in the local coordinate system of $S_{q,r}$, $D_{r,s} = (d_{r,s}, d_{r,s})$. We will define the numbers $d_{r,s}$ in such a way that the following holds:

(a) $\forall r, s$: $\operatorname{card}(D \cap S_{r,s}) = 1$,
(b) $\operatorname{pr}_X D$ is dense open in X and $\operatorname{pr}_Y D$ is dense open in Y.

Let us consider first the uncountable family of squares $S_{r,0}$ lying in the first "row". In the first square $S_{0,0}$ of this family pick $D_{0,0} = (d_{0,0}, d_{0,0}) = (0, 0)$, in the local coordinate system of $S_{0,0}$. Thus $D_{0,0}$ is the lower, left corner. As we increase r, keeping $s = 0$, the point $D_{r,0}$ is gradually moving upwards along the main diagonal of each square until it hits $(1, 1)$. More precisely, we put $d_{r,0} = r$, $r \in [0, 1]$. Now fix $s_0 \in (0, 1]$ and consider the corresponding row of the squares. Put

$$d_{r,s_0} = \begin{cases} r + s_0, & \text{if } r + s_0 \leqslant 1, \\ (r + s_0) - 1, & \text{if } r + s_0 > 1. \end{cases}$$

Thus D_{0,s_0} is a point from the diagonal of S_{0,s_0} different from $(0, 0)$ (in local coordinates). As we increase r, keeping s_0 fixed, the point D_{r,s_0} is gradually moving upwards along the main diagonal of each square until it hits $(1, 1)$. Then it falls down-left and starts growing from right outside of $(0, 0)$ until it reaches its starting position.

Now we are going to define $g : X \times Y \to \mathbb{R}$ by defining its restriction $g_{r,s}$ to each square $S_{r,s}$ as follows (we use local coordinates):

$$g_{r,s}(x, y) = \begin{cases} \frac{2(x-d_{r,s})(y-d_{r,s})}{(x-d_{r,s})^2+(y-d_{r,s})^2}, & \text{if } (x, y) \neq (d_{r,s}, d_{r,s}), \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $g_{r,s}$ is continuous on $S_{r,s}$, except for the point $D_{r,s}$.

It follows from the construction that $C(g)$ contains neither $A \times Y$ nor $X \times B$ for any nonempty set $A \subset X$ or nonempty set $B \subset Y$.

Comment. A somewhat less involved example, one "column" only, was designed by Jack B. Brown (see [1, Example 6.14, p. 313]) to answer in *the negative*, questions by A. Alexiewicz, W. Orlicz [2] and J.P.R. Christensen [3] whether the assumption that *both* spaces X and Y are complete metric, suffices in Namioka-type theorems. In other words, there are complete metric spaces X and Y and a separately continuous function $f : X \times Y \to \mathbb{R}$ such that there is *no* G_δ set $A \subset X$ such that $A \times Y \subset C(f)$; in fact the largest such a set is empty.

References

[1] Z. Piotrowski, Separate and joint continuity, Real Anal. Exchange 11 (1985–86) 293–322.

[2] A. Alexiewicz, W. Orlicz, Sur la continuite et la classification de Baire des fonctions abstraites, Fund. Math. 35 (1948) 105–126.

[3] J.P.R. Christensen, Joint continuity of separately continuous functions, Proc. Amer. Math. Soc. 82 (1981) 455–461.

Further reading

[1] M. Henriksen, R.G. Woods, Separate versus joint continuity: A tale of four topologies, Topology Appl. 97 (1999) 175–205.