

On Semi-Homeomorphisms.

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Sunto. — *Si studiano le relazioni tra omeomorfismi e semi-omeomorfismi nel senso di Crossley e Hildebrand.*

1. — Introduction.

Several authors have considered some connections between Hausdorff or regular spaces and semi-continuity (known also as quasi-continuity, c.f. [2] and [11]) or somewhat continuity—see [1], [12], [13], [15] and [16]. Also semi-homeomorphisms were studied—see [3], [4], [7] and [12].

The main aim of this paper is to show that if $f: X \rightarrow Y$ is a closed function from a regular space X to a regular and dense in itself space Y , then f is open and continuous if and only if f is pre-semi-open and irresolute (see Theorem 14).

Some conditions concerning X and/or Y are investigated in the paper under which the hypothesis that f is closed can be omitted in the above theorem.

The paper also contains examples which prove the essentiality of some assumptions. An answer is given to T. Neubrunn's question concerning semi-homeomorphisms.

Functions investigated in this paper are not necessarily assumed to be continuous; a space means a topological space. We use the terminology as in [4] and [5]. In particular, all kinds of spaces related to the compactness (like locally compact spaces etc.) are assumed to be Hausdorff. Likely, regular and normal spaces are assumed to be T_1 .

2. — Regular spaces and functions.

First we recall some definitions:

A subset A of a space X is said to be semi-open (see [10], Def. 1, p. 36) if there exists an open set U in X such that $U \subset A \subset \bar{U}$.

LEMMA 1 (see [14], Theorem 1, p. 216). — *If a function $f: X \rightarrow Y$ is almost open and semi-continuous then it is irresolute.*

COROLLARY 2. — *If a function $f: X \rightarrow Y$ is open and continuous then it is irresolute.*

LEMMA 3 ([12]). — *If a function $f: X \rightarrow Y$ is continuous and somewhat open then it is pre-semi-open.*

COROLLARY 4. — *If a function $f: X \rightarrow Y$ is open and continuous then it is pre-semi-open.*

LEMMA 5 ([13], Lemma 1, p. 132). — *If U is an open set and A is a semi-open one then $A \cap U$ is semi-open.*

LEMMA 6 ([9], Theorem 1, p. 86). — *A function $f: X \rightarrow Y$ is continuous if and only if for each net $\{S_n, n \in D\}$ (where D denotes a directed set) in X which converges to a point s , the net $\{f(S_n), n \in D\}$ converges to $f(s)$.*

THEOREM 7. — *Let a topological space Y be regular and dense in itself. If a function $f: X \rightarrow Y$ is pre-semi-open and semi-continuous, then it is continuous.*

PROOF. — On the contrary, assume that f is not continuous at a point $x \in X$. By Lemma 6 there exists a net $\{x_n, n \in D\}$ which converges to x and $\{f(x_n), n \in D\}$ does not converge to $f(x)$. Since Y is regular, there exists an open set $G \subset f(X)$ which contains $f(x)$ and open sets $H_n \subset f(X)$, $n \in D$, containing $f(x_n)$ for all $n \in D$, such that $G \cap \bigcup_{n \in D} H_n = \emptyset$. The union $U = \bigcup_{n \in D} \{H_n: n \in D\}$ is open, and since f is semi-continuous, $f^{-1}(U)$ is semi-open. But since $f^{-1}(U)$ contains $\{x_n: n \in D\}$, the point x is a limit point of $f^{-1}(U)$. Then $f^{-1}(U) \cup \{x\}$ is still semi-open. By pre-semi-openness of f the set $f(f^{-1}(U) \cup \{x\}) = f(f^{-1}(U)) \cup \{f(x)\} = U \cup \{f(x)\}$ is semi-open. Since U is disjoint with G , we have $(U \cup \{f(x)\}) \cap G = \{f(x)\}$, and therefore the singleton $\{f(x)\}$ is semi-open, by Lemma 5. But it is obvious that a semi-open singleton is open, so $\{f(x)\}$ is open, to the contrary with our assumptions, because Y is dense in itself.

Since every irresolute function is semi-continuous (see Diagram 2) we have the following corollary to Theorem 7:

COROLLARY 8. — *Let a topological space Y be regular and dense in itself. If a function $f: X \rightarrow Y$ is pre-semi-open and irresolute, then it is continuous.*

Since every pre-semi-open function is somewhat open (see Diagram 1) and every continuous function is semi-continuous (see Diagram 2) then combining Lemma 3 with Theorem 7 we obtain the following characterization of continuous and somewhat open functions.

THEOREM 9. - *Let a topological space Y be regular and dense in itself. A function $f: X \rightarrow Y$ is continuous and somewhat open if and only if f is pre-semi-open and semi-continuous.*

THEOREM 10. - *Let spaces X and Y be regular and let Y be dense in itself. If a function $f: X \rightarrow Y$ is pre-semi-open, irresolute and closed, then f is open.*

PROOF. - We can assume that the image space Y does not reduce to a single point, otherwise the theorem trivially holds. It follows from Corollary 8 that f is continuous.

Now Y is non-degenerate and Hausdorff, so for every $y \in Y$ there exist $x \in Y \setminus \{y\}$ and open sets H_x and H_y , such that $y \in H_y$, $x \in H_x$ with $H_x \cap H_y = \emptyset$. Thus $Y_y \subset Y \setminus H_x$, whence $\overline{H_x} \subset \overline{Y \setminus H_x} = Y \setminus H_x$ and therefore $H_x \subset Y \setminus \overline{H_y}$. So we see that

$$(*) \quad Y \setminus H_y \neq \emptyset \quad \text{for every } y \in Y.$$

Suppose that f is not open. Then there exists an open set $G \subset X$ such that $f(G)$ is not open. But

$$\begin{aligned} G &= G \cap X = G \cap f^{-1}(Y) = G \cap f^{-1}\left(\bigcup_{y \in Y} H_y\right) = \\ &= G \cap \bigcup_{y \in Y} f^{-1}(H_y) = \bigcup_{y \in Y} (G \cap f^{-1}(H_y)). \end{aligned}$$

Therefore $f(G) = \bigcup_{y \in Y} f(G \cap f^{-1}(H_y))$. By continuity of f , the sets $G \cap f^{-1}(H_y)$ are open. Since $f(G)$ is not open, there is $y_0 \in Y$, such that $f(G \cap f^{-1}(H_{y_0}))$ is not open. Put $V = G \cap f^{-1}(H_{y_0})$. Thus $f(V) = f(G) \cap H_{y_0} \subset H_{y_0}$, whence $Y \setminus f(V) \supset Y \setminus H_{y_0} \neq \emptyset$ by (*). Thus $f(V)$ is not dense.

The set $f(V)$ being not open, there is a point $x \in V$ such that $f(x)$ does not belong to $\text{Int } f(V)$. By regularity of X , there is a closed neighborhood F of x such that $F \subset V$. Now, by the assumption of the theorem, $f(F)$ is closed, and we see that $U = Y \setminus f(F)$ is a non-empty, open set.

Since $f(x)$ is not an interior point of $f(V)$, thus of $f(F)$, hence $U \cup \{f(x)\}$ is semi-open. The function f is irresolute, and there-

fore the set $A = f^{-1}(U \cup \{f(x)\})$ is semi-open. Since $\text{Int } F$ is open, it follows from Lemma 5, that $A \cap \text{Int } F$ is semi-open. Now, f being pre-semi-open, so the set $f(A \cap \text{Int } F)$ is semi-open. But

$$\begin{aligned} f(A \cap \text{Int } F) &= f(f^{-1}(U \cup \{f(x)\}) \cap \text{Int } F) = \\ &= (U \cup \{f(x)\}) \cap f(\text{Int } F) = \{f(x)\}, \end{aligned}$$

we see that the singleton $\{f(x)\}$ is semi-open and thus it is open. But, Y being dense in itself, this is a contradiction finishing the proof of Theorem 10.

The following examples show that the requirements of irresoluteness and of pre-semi-openness of f in the above theorem are essential.

EXAMPLE 11. - Let $f: [0, 1] \rightarrow [0, 1]$ be a function as follows:

$$f(x) = \begin{cases} x + \frac{1}{2}, & \text{for } 0 < x < \frac{1}{2}, \\ -2x + 2, & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

The function f manifestly is pre-semi-open and closed; it is not irresolute, since $f^{-1}([0, \frac{1}{2}])$ is not semi-open, it is also not open, $f([0, \frac{1}{2})) = [\frac{1}{2}, \frac{3}{4})$.

EXAMPLE 12 ([6], Chapter 12, Example 15). Let X be the unit circle $\{(x, y) | x^2 + y^2 = 1\}$ with the topology induced from the plane. Let Y be the interval $[0, \pi)$ with the topology induced from the reals. And let $f: X \rightarrow Y$ be defined as follows:

$$f(\cos \alpha, \sin \alpha) = \begin{cases} 0 & \text{if } 0 < \alpha < \pi, \\ \alpha - \pi & \text{if } \pi < \alpha < 2\pi. \end{cases}$$

It is easy to verify that the function f is irresolute and closed without being pre-semi-open and open.

Concerning the other assumptions of Theorem 10, both treating on the spaces X and Y and the function f , the author does not know if they are essential or not. However, one can conjecture that every pre-semi-open and irresolute function $f: X \rightarrow Y$ (where X and Y are regular and Y is dense in itself) is closed. It is not so, even in case when X and Y are Euclidean spaces.

The following example shows that the assumption that f is closed is independent from the other assumption on f in Theorem 10.

EXAMPLE 13. - Let X and Y be the Euclidean plane and the real line, respectively. Let $f: X \rightarrow Y$ be the projection from X

onto Y , i.e., $f(x, y) = x$. It is easy to see that f is pre-semi-open and irresolute while it is not closed. Unfortunately it is also an open function and thus this example does not show the essentiality of the assumption (in Theorem 10) that f is closed.

Combining Corollaries 2 and 4 with Corollary 8 and Theorem 10 we can state the main result of the paper:

THEOREM 14. — *Let spaces X and Y be regular and Y dense in itself. Let a function $f: X \rightarrow Y$ be closed. Then f is open and continuous if and only if f is pre-semi-open and irresolute.*

We recall the following

LEMMA 15 ([8], Lemma 3, p. 187). — *Let a space X be countably compact and let a space Y be Hausdorff and first countable. If a function $f: X \rightarrow Y$ is continuous, then f is closed.*

Now we will show some consequences of Theorem 14.

THEOREM 16. — *Let f be a function from a space X into a space Y . If:*

(1) *X is regular, countably compact and Y is regular, first countable, dense in itself, or*

(2) *X is locally compact and Y is regular and dense in itself, then: f is open and continuous if and only if f is pre-semi-open and irresolute.*

PROOF. — The « only if » implication is a consequence of Corollaries 2 and 4 (without any assumptions concerning the spaces X and Y , thus for both cases (1) and (2)).

For the « if » implication, the continuity of f follows from Corollary 8 (also for both cases (1) and (2)). Concerning the openness of f , we consider cases (1) and (2) separately.

In case (1), since the function f is continuous (as it was just proved), we conclude from Lemma 15 that it is closed, and hence we are done by Theorem 10.

In case (2), to show that the (continuous) function f in matter is open, let us consider a compact neighborhood F of a point $x \in X$. Thus $f(F)$ is compact, and since Y is Hausdorff, $f(F)$ is closed. The rest part of the proof is exactly the same as the corresponding part of the proof of Theorem 10.

Theorem 16 stays valid if the words « X is regular, countably compact » in (1) are replaced by « X is normal, pseudocompact » or « X is regular, sequentially compact ». For arguments of these facts—see [5], Th. 3.10.21, p. 263 and Th. 3.10.30, p. 266.

3. - Relations to semi-homeomorphisms.

A bijective pre-semi-open and irresolute function is called semi-homeomorphism (in the sense of Crossley and Hildebrand—see [4], Definition 1.3, p. 236).

Among these results of [4] are the following: The property of being a Hausdorff space and of being a space of the first (second) category are semi-topological properties, i.e., properties which are preserved under semi-homeomorphisms, while regularity, normality, compactness and metrizability are not. Some of these results, in particular ones concerning invariant properties with respect to semi-homeomorphisms have been generalized to a wider class of functions, namely to somewhat homeomorphisms = bijective, somewhat open and somewhat continuous functions—see [7].

In connection with these results it is natural to ask under what conditions on X and Y and, perhaps, on $f: X \rightarrow Y$ a semi-homeomorphism f is a homeomorphism.

N. Biswas [3] defined generalized homeomorphisms (he also calls them semi-homeomorphisms) as one-to-one, semi-open and continuous functions between X and Y . T. Neubrunn has proved [12] that in general topological spaces semi-homeomorphisms in the sense of Crossley and Hildebrand need not be semi-homeomorphisms in the sense of Biswas. Thus the question arises under what assumptions on X and Y and—perhaps—on $f: X \rightarrow Y$, a semi-homeomorphism f in the sense of Crossley and Hildebrand is a semi-homeomorphism in the sense of Biswas.

The following theorem which is a consequence of Theorem 14 gives an answer to the first of the questions

THEOREM 17. - *Let spaces X and Y be regular and Y be dense in itself. Let a function $f: X \rightarrow Y$ be closed. Then f is a homeomorphism if and only if f is a semi-homeomorphism in the sense of Crossley and Hildebrand.*

The hypothesis that f is closed can be omitted in the above theorem provided that some additional assumptions are made on X and/or Y , e.g., X is countably compact and Y is first countable, or X is locally compact (c.f. here, Theorem 16).

From Corollary 8 and Diagram 1 it follows the following answer to the second of the questions. Namely we have

THEOREM 18. - *Let a topological space Y be regular and dense in itself. If $f: X \rightarrow Y$ is a semi-homeomorphism in the sense of Crossley and Hildebrand then f is a semi-homeomorphism in the sense of Biswas.*

In the last sentence of [12] T. Neubrunn asks whether there exists a semi-homeomorphism in the sense of Biswas without being a semi-homeomorphism in the sense of Crossley and Hildebrand. The following example gives an answer to this question.

EXAMPLE 19. - Let $X = Y = \{a, b, c, d\}$. Let \mathcal{O}_1 and \mathcal{O}_2 denote the topologies for X and Y , respectively, such that

$$\mathcal{O}_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\} \text{ and } \mathcal{O}_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}.$$

Let $f: (X, \mathcal{O}_1) \rightarrow (Y, \mathcal{O}_2)$ be the identity function. It is easy to verify that f is continuous and semi-open but not irresolute since $\{a, c\}$ is semi-open in Y , while it is not semi-open in X .

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