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Mibu-type theorems.

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ABSTRACT. *A Mibu-type theorem* is any result in which a class of not necessarily continuous y-sections f_y is specified, so that for any $f : [0,1] \times [0,1] \to \mathbb{R}$ having all x-sections f_x continuous, the set $C(f)$ of point of continuous, the set $C(f)$ of point of continuity of f is a dense, G_δ subset of $[0,1] \times [0,1]$.

In this paper a different proof of a Mibu-type theorem is provided. For the reader's convenience though, we quote the most needed and relevant definitions and theorems.

We illustrate the presentation of the results with a few examples showing that some of the theorems are best possible. Two questions are posed.

§1. INTRODUCTION

Let us start from the following:

(*) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function having all x-sections f_x continuous. What must all y-sections f_y be in order for f to have a dense G_δ set $C(f)$ of points of continuity?

Of course, if all y-sections are continuous, then by a theorem of R. Baire [Ba] such an f is of first class of Baire, and as such it has a dense G_δ set $C(f)$, see [P2] for further generalizations.

But what if we weaken, a bit, the assumptions pertaining to the sections?

As an example, [P1] shows one cannot relax these assumptions too much, since there is a function $f : [0,1] \times [0,1] \to \mathbb{R}$ having: all but countably many x-sections (resp. y-sections) continuous, whereas these countably many x-sections (resp. y-sections) have only *finitely* many points of discontinuity (thus all x-sections and all y-sections are being of first class) while the set $C(f) = \phi$.

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Now, assume $f : [0,1] \times [0,1] \to \mathbb{R}$ has all of its x-sections f_x continuous and all of its y-sections f_y of first class of Baire. What is the set $C(f)$?

There is no general theory that would provide an "automatic" answer. The celebrated theorem of Baire, later generalized by H. Lebesgue, K. Kuratowski and D. Montgomery asserts that such an f is of second class of Baire. But second class of Baire functions may have empty set of points of continuity(!); take the "salt & pepper" function, $f(x) = 0$, if x is rational $f(x) = 1$, if x is irrational.

Before we answer the above specific question as well as provide a few answers to a general problem (*) mentioned at the beginning of the Introduction, we need a couple of defintions.

§2. Definitions.

A space X is called *Baire* if every open nonempty subset of X is of second category.

If $A \subset X$ and $\mathcal{U}$ is a collection of subsets of X, then $st(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \phi\}$.

For $x \in X$, we write $st(x,\mathcal{U})$ instead of $st(\{x\},\mathcal{U})$.

A sequence $\{G_n\}$ of open covers of X is called a *development of* X if for each $x \in X$ the set $\{st(x, G_n) : n \in N\}$ is a base at x.

A *developable space* is a space which has a development[1]

A *Moore space* is a regular developable space.

A function $f : X \to Y$ is called *quasi-continuous at a point* $x \in X$ if for each open sets $A \subset X$ and $H \subset f(X)$, where $x \in A$ and $f(x) \in H$ we have $A \cap Int f^{-1}(H) \neq \phi$.

A function $f : X \to Y$ is called *quasi-continuous* if it is quasi-contimuous at each point of X.[2]

Given spaces X, Y and Z, a function $f : X \times Y \to Z$ is said to be *quasi-continuous at* $(p,q) \in X \times Y$ *with respect to the variable* y if for every neighborhood N of $f(p,q)$ and for every neighborhood $U \times V$ of (p,q) there exists a neighborhood V^1 of q with $V^1 \subset V$ and a nonempty open $U^1 \subset U$ such that for all $(x,y) \in U^1 \times V^1$ we have $f(x,y) \in N$. If f is quasi-continuous with respect to the variable y at every point of its domain, we say that f is *quasi-continuous with respect to* y.

Given a metric space M, a function $f : X \to M$ is called *of first class (in the sense of G. Debs)* if for every $\epsilon > 0$, for every nonempty subset $A \subset X$, there is a nonempty set U, open in A, such that $diam(f(U)) \leq \epsilon$.

For "nice" spaces, say $X = Y = \mathbb{R}$, being the domain and the range of f, respectively we have the following diagram, where "$\longrightarrow$" denotes the inclusion:

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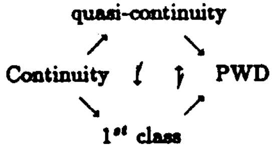
[1]Every metric space is developable; take the family B_ϵ of balls of diameter less than ϵ as a development.

[2]See [Ba] p. 95, see also [HT] and [Tr].



where PWD stands for the pointwise discontinuity; a function f is *pointwise discontinuous* if $C(f)$ is dense.

§3. Some Mibu-type theorems.

We are now ready to give some answers to the spectacular question (*) mentioned at the beginning of the Introduction.

Due to the fact, that Y. Mibu [Mi] was *the first* to answer this question, for f_y's not being necessarily continuous, any positive answer to (*) we shall call a *Mibu-type theorem*.

Consider the following four theorems[3]

Theorem 1. [Mi] – Mibu's First Theorem.

Let X be first countable, Y be Baire and such that $X \times Y$ be Baire. Given a metric space M. If $f : X \times Y \to M$ is separately continous, then $C(f)$ is a dense G_δ subset of $X \times Y$.

Theorem 2. [Mi] – Mibu's Second Theorem.

Let X be second countable, Y be Baire and such that $X \times Y$ is Baire. Given a metric space M. If $f : X \times Y \to M$ has:

a) all x-sections f_x PWD and;

b) all y-sections f_y continuous; then $C(f)$ is a dense G_δ subset of $X \times Y$.

Theorem 3. [De] – Deb's Theorem.

Let X be first countable, Y be a special α-favorable space (thus Baire), $X \times Y$ be Baire. Given a metric space M.

If $f : X \times Y \to M$ has:

a) all x-sections f_x of first class (in the sense of G. Debs) and;

b) all y-sections f_y continuous; then $C(f)$ is a dense G_δ subset of $X \times Y$.

Theorem 4. [P1], Theorem B.

Let X be first countable, Y be Baire and Z be Moore. If $f : X \times Y \to Z$ has:

a) all x-sections f_x quasi-continuous and;

[3]Theorem 4 implies Theorem 1

b) all y-sections f_y continuous, then $C(f)$ is a dense G_δ subset of $\{x\} \times Y$; thus $C(f)$ is a dense, G_δ subset of $X \times Y$.

In this paper we shall give a shorter proof of the above Theorem 4; rather than using, ad hoc, a Banach-Mazur game, we shall derive this result from some properties of a generalized oscillation function Ω.

§4. Generalized oscillation function and quasi-continuity.

If $f : X \to Y$ is a function and $\mathcal{P}$ is an open cover of the space Y, then we set:[4]

$\Omega(f, \mathcal{P}) = \{x \in X$: there is an open neighborhood U of x and a member V of $\mathcal{P}$ such that $f(U) \subset V\}$.

If Y is a metric space and $\mathcal{P}_\varepsilon$ is the family of all balls of diameter less than ε, then $\Omega(f, \mathcal{P}_\varepsilon) = \omega(f, \mathcal{P}_\varepsilon)$ is the set of all points where the oscillation of f does not exceed ε.

The sets of the form $\Omega(f, \mathcal{P})$ are open.

Lemma 1. Let X and Y be spaces and let $\{\mathcal{P}_n\}$ be a development for Z. If $f : X \times Y \to Z$ is quasi-continuous with respect to x, then:

$\Omega(f, \{\mathcal{P}_n\})$ is dense in $\{x\} \times Y$ for every $x \in X$.

Proof. Let $x_0 \in X$ be arbitrary and let V be a nonempty open subset of Y. Now, let y_0 be an arbitrary element of V. Since Z has a countable development $\{\mathcal{P}_n\}$, there is a local countable base at every point of Z; in particular, take $\{G^n\}$ at $f(x_0, y_0)$. Take G^1. Now by the quasi-continuity of f with respect to x for every neighborhood $U^* \times V^*$ of (x_0, y_0) (here, we may assume $V^* = V$), there exists a neighborhood U^1 of x, with $U^1 \subset U$ and a nonempty open $V^1 \subset V$ such that $f(U^1 \times V^1) \subset G^1$.

Now, let us recall that $\Omega(f, \{\mathcal{P}_n\}) = \{(x, y) \in X \times Y$: there is an open neighborhood $U_\Omega \times V_\Omega$ of (x, y) and a member P of $\mathcal{P}_n$ such that $f(U_\Omega \times V_\Omega) \subset P\}$. Since $\{G^n\}$ is a local base at $f(x_0, y_0)$ we may assume $P = G^1$. We can also assume $U_\Omega = U^1$ and $V_\Omega = V^1$. Clearly, $f(U_\Omega \times V_\Omega) = f(U^1 \times V^1) \subset G^1 = P$.

Further, $\Omega(f, \{\mathcal{P}_n\}) = U^1 \times V^1$. Observe that $\Omega(f, \{\mathcal{P}_n\}) \cap (\{x_0\} \times V) = (U^1 \times V^1) \cap (\{x_0\} \times V) = \{x_0\} \times V^1 \neq \phi$. Since V is an arbitrary nonempty open subset of Y this shows the density of $\Omega(f, \{\mathcal{P}_n\})$ in $\{x\} \times Y$. □

Lemma 1 [Sz]. Let $\{\mathcal{P}_n\}$ be a development for Y and assume $\Omega(f, \mathcal{P}_n)$ is dense in X, for each $n \in N$. Then the function f is continuous at each point of a residual subset of X.

Corollary 1. ([P1], Theorem A) Let X be a space Y be Baire and let $\{\mathcal{P}_n\}$ be a development for Z. If $f : X \times Y \to Z$ is quasi-continuous with respect to x, for all $x \in X$, then the set

[4] The original idea is due to A. Szymański [Sz], although some similar notions have been used earlier, see [Ew] or [Is].

$C(f)$ of points of continuity of f is residual in the sets of the form $\{x\} \times Y$, i.e. is a dense, G_δ subset in the sets of the form $\{x\} \times Y$.

Now, in view of:

Lemma 3. ([LP], Lemma 2) Let X be first countable, Y be Baire and Z regular. If $f : X \times Y \to Z$ has all of its x-sections f_x quasi-continuous and has its y-sections f_y continuous, with the exception of a first category set, then f is quasi-continuous with respect to x.

We have the following statement which *slightly* generalizes our Theorem 4.

Statement. Let X be first countable, Y be Baire and Z regular. If $f : X \times Y \to Z$ has all of its x-sections f_x quasi-continuous and has its y-sections f_y continuous with the exception of a first category set, then the set $C(f)$ of continuity points of f is a dense G_δ in the sets of the form $\{x\} \times Y$.

§5. Examples and open questions.

The following (routine) Example 1 shows that "quasi-continuity with respect to x" cannot be replaced by "quasi-continuity" in Corollary 1.

Example 1. Let $f : (0,1) \times (0,1) \to \mathbb{R}$ be defined by $f(x,y) = 1$ if $(0 < x < \frac{1}{2}$ and $0 < y < 1)$ or $(x = \frac{1}{2}$ and y is rational in $(0,1))$, and $f(x,y) = 0$. Clearly, f is quasi-continuous, however there *is* an $x(= \frac{1}{2})$ such that $C(f) \cap (\{x\} \times Y) = \phi$.

Remark 1. There is an important link between continuity points in sets of type $\{x\} \times Y$[5] and some associated mapping.

Namely, any point of the section $\{x\} \times Y$ is a continuity point of $f : X \times Y \to [-1,1]$ if and only if x is a continuity point for the associated mapping:

$F : X \to C(Y)$ defined by

$F(x)(y) = f(x,y)$, with respect to norm-topology on $C(Y)$, see [Ch]; $C(Y)$ stands for the Banach space of continuous functions on Y.

The following two Examples show that the assumptions that X is "*first countable*" and that Z is "*Moore*" are, in a sense, indispensable.

Example 2. ([T1]) Let Y and Z denote the closed unit interval $I = [0,1]$ and let X be the space $C_p(I,I)$ of continuous functions from I into I, equipped witht he pointwise topology. Then $f : X \times Y \to Z$ given by $f(x,y) = x(y)$ is separately continuous which is discontinnous at *every* point of $X \times Y$.

It is worth noting that $C_p(I,I)$ is a Tychonoff space having a countable network [E. Michael (1966)] and as such it is hereditarily Lindelöf and hereditarily separable. $C_p(I,I)$

[5] See Theorem 4.

is not a Frechet space and thus not first countable [R. A. McCoy (1980) and J. Gerlits (1983)].

The following Example 3 shows the necessity that the range space Z is Moore.

Example 3. [Ch] Let $X = Y = [-1, 1]$ and let Z be the space of mappings from $[-1, 1]^2$ into $[-1, 1]$ equipped with the pointwise topology. Thus $F(x, y)$ is a function of $(a, b) \in [-1, 1]^2$ given by

$$F(x,y)(a,b) = \begin{cases} \dfrac{2(x-a)(y-b)}{(x-a)^2+(y-b)^2}, & \text{if this quotient is defined} \\ 0, & \text{otherwise} \end{cases}$$

F is separately continuous, but not (jointly) continuous at any point. Z is a "large" compact Hausdorff space.

We were unable to answer the following two problems:

Problem 1. Let X be first countable, Y be Baire and such that $X \times Y$ is Baire. Given a Moore space (or even metric space) Z. Assume $f : X \times Y \to Z$ has:

a) all x-sections f_x PWD and;

b) all y-sections f_y continuous.

Must f be PWD, i.e. $C(f)$ is a dense G_δ subset of $X \times Y$?

Problem 2. Same as Problem 1, except for, assume X is "compact Hausdorff" instead of "first countable".

Remark 2. Answers in positive, to Problem 1 would be strong generalizations of some of the above Theorems.

Symbolically, this would mean: "PWD" × "Continuity" = "PWD"

Remark 3. Answers in positive, to Problem 2, would solve an outstanding problem of M. Talagrand [T2] whether every real-valued, separately continuous function from a Cartesian product of a Baire space and a compact Hausdorff space has a nonempty set $C(f)$ of points of continuity.

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