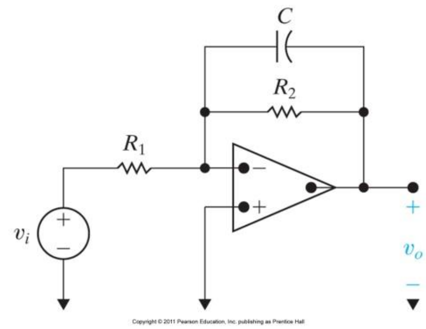
Chapter 15: Active Filter Circuits

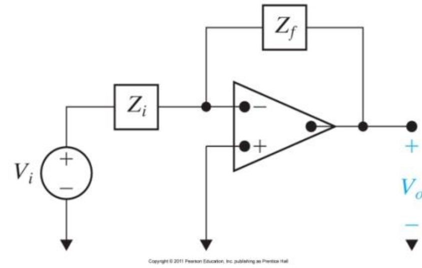
15.1 First-Order Low-Pass and High-Pass Filters

For the circuit, when the frequency changes only the impedance of the capacitor is affected.



At low frequency the capacitor is open and the gain of the circuit is $-{}^{R_2}/_{R_1}$

At high frequency the capacitor acts as a short and grounds the input, thus a low-pass filter.



Replacing the first circuit with an equivalent general op amp circuit and analyzing our example:

$$Z_i = R_1;\ Z_f = R_2 \parallel C$$

Writing the transfer function

$$H(s) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i} = -\frac{R_2 \parallel \frac{1}{sC}}{R_1} = -\frac{\frac{R_2}{sC}}{R_1\left(R_2 + \frac{1}{sC}\right)} = -\frac{\frac{R_2}{C}}{R_1\left(sR_2 + \frac{1}{C}\right)}$$

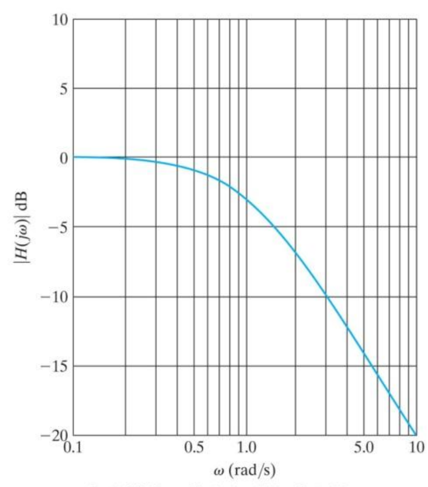
$$= -\frac{\frac{R_2}{R_1}\left(\frac{1}{R_2C}\right)}{s + \frac{1}{R_2C}} = -K\frac{\omega_c}{s + \omega_c}$$

Where

$$Gain:\quad K = \frac{R_2}{R_1} \qquad and \qquad Cutoff\ Frquency:\quad \omega_c = \frac{1}{R_2C}$$

Note: With an op amp the gain and cut-off frequency can be determined independently

Frequency Response Plots:



Bode Plots: {See Appendix E}

- Plotted on logarithmic axis – allowing more frequencies to be visible
- Plotted in decibels (dB) instead of magnitude {See Appendix D}

  Converting to decibel

$$A_{dB} = 20\log_{10}|H(j\omega)|$$

Since A is a signed value and $|H|$ is not:

When $A_{dB} < 0;\quad 0 \le |H| < 1$

$A_{dB} > 0;\quad |H| > 1$

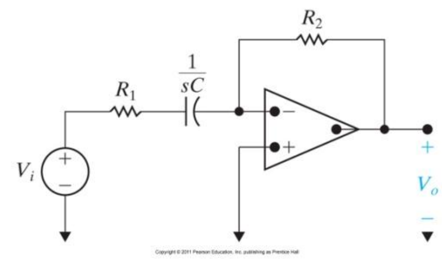
$A_{dB} = 0;\quad |H| = 1$

Analyzing at the cut-off frequency

$$A_{dB} = 20\log_{10}\frac{1}{\sqrt{2}} = -3dB$$

Therefore the cutoff frequency can be seen where the maximum magnitude in decibels is reduced by 3 dB.

The next figure represents a first order high-pass filter.



$$H(s) = -\frac{Z_f}{Z_i} = -\frac{R_2}{R_1+\frac{1}{sC}} = -K\frac{s}{s+\omega_c}$$

Where

$$Gain:\quad K = \frac{R_2}{R_1} \qquad and \qquad Cutoff\ Frquency:\quad \omega_c = \frac{1}{R_1 C}$$

Note: the transfer functions for both the low-pass and high-pass active filters are the same as the transfer functions for the passive filters discussed in the previous chapter

15.2 Scaling

2-types:

Magnitude scaling: multiple the impedances at a given frequency by scale factor k_m

$$R' = k_m R; \quad L' = k_m L; \quad C' = {}^{C}/_{k_m}$$

Where k_m is any positive real number less than or greater than 1

Frequency scaling: change the circuit such that at a new frequency the impedances are the same as the original frequency using scaling factor k_f.

$$R' = R; \quad L' = {}^{L}/_{k_f}; \quad C' = {}^{C}/_{k_f}$$

A circuit can be scaled simultaneously for both magnitude and frequency

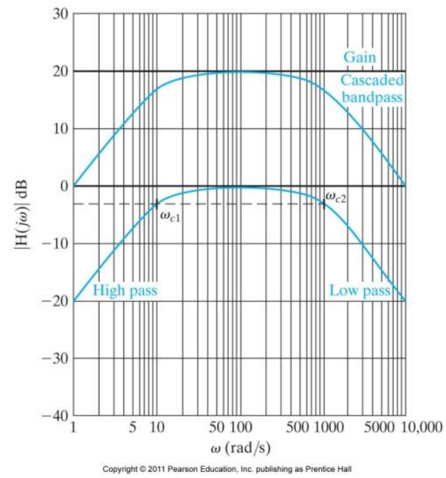
$$R' = k_m R; \quad L' = \frac{k_m}{k_f}L; \quad C' = \frac{1}{k_m k_f}C$$

Use of Scaling in Design

1. Select $\omega_c = 1$ for low- or high-pass filter OR $\omega_o = 1$ for bandpass or bandreject filters
2. Select a 1F capacitor and calculate the values for the resistors that give the 1 rad/s frequency above
3. Use scaling to determine more realistic values for the resistor and capacitors at the desired frequency

15.3 Op Amp Bandpass and Bandreject Filters

A Bandpass filter can be considered to a combination of three separate components:



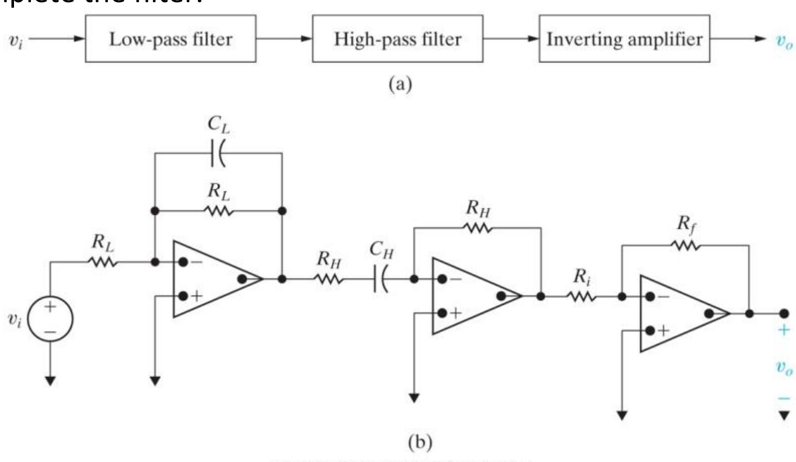
1. A unity-gain low-pass filter whose cut-off frequency is ω_{c2}, the larger of the two cut-off frequencies
2. A unity-gain high-pass filter whose cut-off frequency is ω_{c1}, the smaller of the two cut-off frequencies
3. A gain component to provide the desired level of gain in the pass band.

These items can be cascaded in sereies; where $\omega_{c1} < \omega_{c2}$and are called a broadband bandpass filter.

A broadband bandpass filter is defined

$$\frac{\omega_{c2}}{\omega_{c1}} \geq 2$$

The circuitcan be represented as a block diagram ech illistrating the indivdual circuit required to complete the filter.



The transfer function of the broadband Bandpass filter is the product of the transfer functions of the three cascaded components.

$$H(s) = \frac{V_o}{V_i} = \left(\frac{-\omega_{c2}}{s+\omega_{c2}}\right)\left(\frac{-s}{s+\omega_{c1}}\right)\left(\frac{-R_f}{R_i}\right) = \frac{-K\omega_{c2}s}{(s+\omega_{c2})(s+\omega_{c1})}$$

$$= \frac{-K\omega_{c2}s}{s^2 + (\omega_{c1}+\omega_{c2})s + \omega_{c1}\omega_{c2}}$$

To make this equation match of standard form determined in chapter 14 $\omega_{c2} \gg \omega_{c1}$

$$H(s) = \frac{-K\omega_{c2}s}{s^2 + \omega_{c2}s + \omega_{c1}\omega_{c2}}$$

Determine the values of R_L and C_L in the low-pass filter to meet the upper cutoff frequency

$$\omega_{c2} = {}^{1}\!/_{R_L C_L}$$

Determine the values of R_H and C_H in the high-pass filter to meet the lower cutoff frequency

$$\omega_{c1} = {}^{1}\!/_{R_H C_H}$$

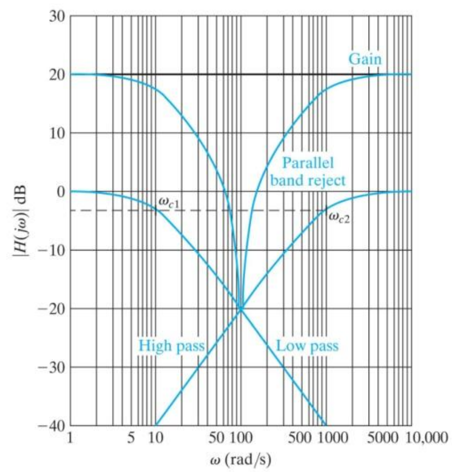
Evaluate the magnitude of the transfer function at the center frequency: $\omega_o = \sqrt{\omega_{c1}\omega_{c2}}$

$$|H(j\omega_o)| = \frac{-K\omega_{c2}(j\omega_o)}{(j\omega_o)^2 + \omega_{c2}j\omega_o + \omega_{c1}\omega_{c2}} = \frac{K\omega_{c2}}{\omega_{c2}} = K$$

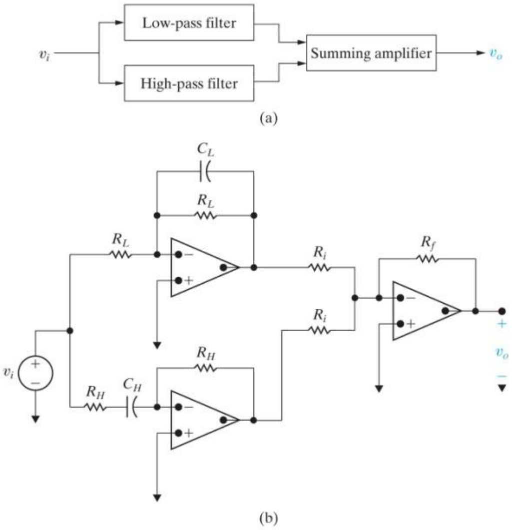
$$Gain\ of\ an\ inverting\ amplifier\ is\ \ |H(j\omega_o)| = K = \frac{R_f}{R_i}$$

A Bandreject filter can also be considered to a combination of three separate components:

1. A unity-gain low-pass filter whose cut-off frequency is ω_{c1}, the smaller of the two cut-off frequencies
2. A unity-gain high-pass filter whose cut-off frequency is ω_{c2}, the larger of the two cut-off frequencies
3. A gain component to provide the desired level of gain in the passbands.



Unlike the Bandpass these items are not connected in series rather the combine as a parallel connection and a summing junction.



The transfer function:

$$H(s) = \left(\frac{-\omega_{c1}}{s + \omega_{c1}} + \frac{-s}{s + \omega_{c2}}\right)\left(\frac{-R_f}{R_i}\right)$$

$$= \frac{R_f}{R_i}\left(\frac{\omega_{c1}(s + \omega_{c2}) + s(s + \omega_{c1})}{(s + \omega_{c1})(s + \omega_{c2})}\right)$$

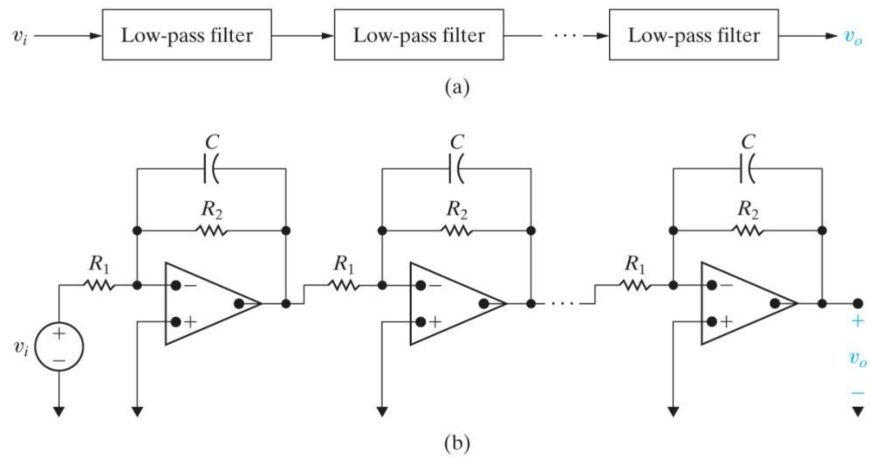
$$= \frac{R_f}{R_i}\left(\frac{s^2 + 2\omega_{c1}s + \omega_{c1}\omega_{c2}}{(s + \omega_{c1})(s + \omega_{c2})}\right)$$

Cutoff Frequencies & Gain

$$\omega_{c1} = \frac{1}{R_L C_L};\ \ \omega_{c2} = \frac{1}{R_H C_H}\ \ and\ K = \frac{R_f}{R_i}$$

15.4 Higher Order Op Amp Filters

By cascading multiple identical low-pass filters together the transition from passband to stopband becomes sharper and closer to an ideal filter.



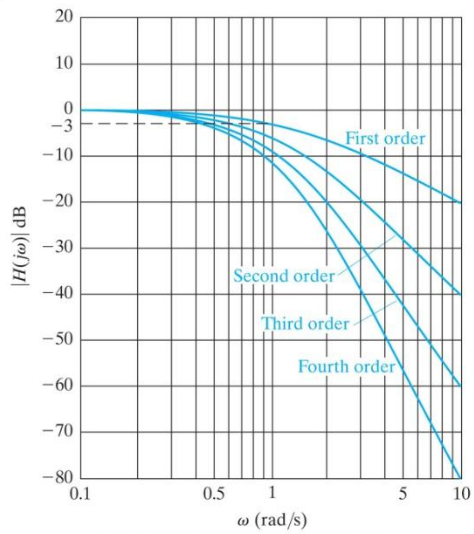
(a)
(b)

In general, an n-element cascade of identical low-pass filters will transition from passband to stopband with a slope of $20n\frac{dB}{dec.}$.

The transfer function for cascaded filters can be determined from multiplying.

$$H(s) = \left(\frac{-1}{s+1}\right)\left(\frac{-1}{s+1}\right)\cdots\left(\frac{-1}{s+1}\right)$$
$$= \frac{(-1)^n}{(s+1)^n}$$

The order of the filter can be determined from the number of poles



As the order increases, the cutoff frequency changes and the use of scaling will be necessary to correct it. First solving for ω_{cn}

$$|H(j\omega_{cn})| = \left|\frac{1}{(j\omega_{cn}+1)^n}\right| = \frac{1}{\sqrt{2}}$$
$$\omega_{cn} = \sqrt{\sqrt[n]{2} - 1}$$

Example 4th order

$$\omega_{c4} = \sqrt{\sqrt[4]{2} - 1} = 0.435$$

To correct the cutoff frequeny

$$k_f = \frac{\omega_c}{0.435}$$

One drawback of the cascading filter is that the gain does not remain constant from zero to the cutoff frequency

$$H(s) = \frac{\omega_{cn}^n}{(s+\omega_{cn})^n}$$
$$|H(j\omega_{cn})| = \frac{\omega_c^n}{\left(\sqrt{\omega^2+\omega_{cn}^n}\right)^n} = \frac{1}{\left(\sqrt{\left({}^{\omega}/_{\omega_{cn}}\right)^2+1}\right)^2}$$

Butterworth Filters

A unity gain Butterworth low-pass filter magnitude

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\omega/\omega_c\right)^{2n}}}$$

- The cutoff frequency is ω_c for all values of n
- If n is large enough the denominator is close to unity
- The exponent of ω/ω_c is always even

Using prototype filters to solve for the transfer function

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega)$$

For $s = j\omega$

$$|H(j\omega)|^2 = H(s)H(-s)$$

And $s^2 = -\omega^2$

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^{2n}} = \frac{1}{1 + (\omega^2)^n} = \frac{1}{1 + (-s^2)^n} = \frac{1}{1 + (-1)^n s^{2n}}$$

$$H(s)H(-s) = \frac{1}{1 + (-1)^n s^{2n}}$$

<u>Procedure for finding H(s)</u> *(See Example 15.8)*

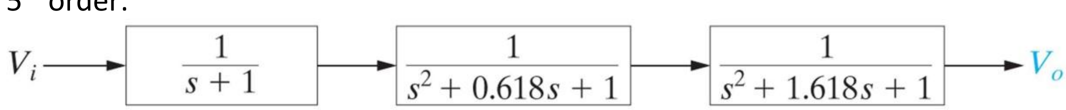
1. Find the roots of the polynomial $\{1 + (-1)^n s^{2n} = 0\}$
2. Assign the left-half plane roots to H(s) and the right-half to H(-s)
3. Combine terms in the denominator of H(s) to form first and second-order factors

TABLE 15.1 Normalized (so that $\omega_c = 1$ rad/s) Butterworth Polynomials up to the Eighth Order

n	nth-Order Butterworth Polynomial
1	$(s + 1)$
2	$(s^2 + \sqrt{2}s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$
5	$(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + \sqrt{2} + 1)(s^2 + 1.932s + 1)$
7	$(s + 1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$
8	$(s^2 + 0.390s + 1)(s^2 + 1.111s + 1)(s^2 + 1.6663s + 1)(s^2 + 1.962s + 1)$

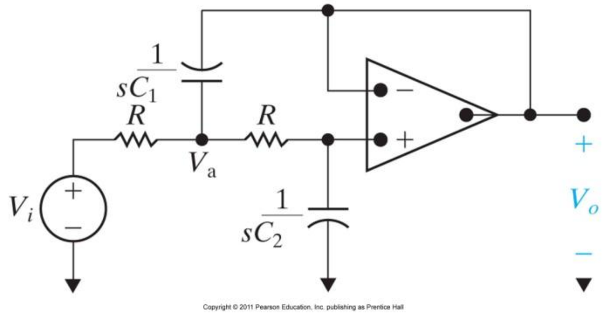
Note: The Butterworth polynomials are products of first- and second-order factors and can be modeled by cascading op amp circuits.

For a 5th order:



All odd-order Butterworth polynomials contain a $\frac{1}{s+1}$ component which can be represented by the prototype low-pass op amp filter discussed earlier.

The second-order op amp filter circuit looks like the one to the right.



$$H(s) = \frac{V_o}{V_i} = \frac{\frac{1}{R^2C_1C_2}}{s^2 + \frac{2}{RC_1}s + \frac{1}{R^2C_1C_2}}$$

Setting R=1

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{1}{C_1C_2}}{s^2 + \frac{2}{C_1}s + \frac{1}{C_1C_2}} = \frac{1}{s^2 + b_1 s + 1}$$

For

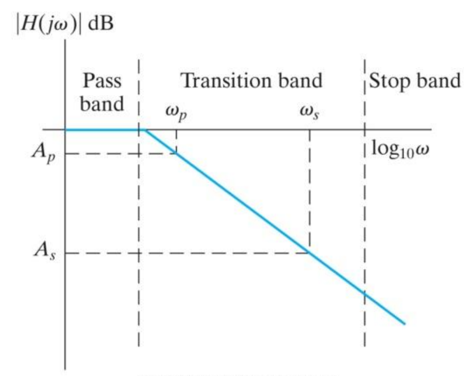
$$b_1 = \frac{2}{C_1} \; and \; 1 = \frac{1}{C_1C_2}$$

This would be the procedure to employ to design an nth-order Butterworth low-pass filter circuit with a cutoff frequency of 1 rad./s and gain of 1.

The order of a Butterworth filter

The higher the order of the filter the closer the filter mimics an ideal filter however the higher the order also means increased circuit components; thus the smallest value of n to meet the design specifications needs to be determined.

From the Bode plot:



$$|H(j\omega)| = A_p = 20\log_{10}\frac{1}{\sqrt{1+{\omega_p}^{2n}}}$$
$$= -10\log_{10}\left(1+{\omega_p}^{2n}\right)$$

$$A_s = 20\log_{10}\frac{1}{\sqrt{1+{\omega_s}^{2n}}} = -10\log_{10}\left(1+{\omega_s}^{2n}\right)$$

Rewriting

$$10^{-0.1A_p} = 1+{\omega_p}^{2n} \;\; and \;\; 10^{-0.1A_s} = 1+{\omega_s}^{2n}$$

Solving for the frequency and creating a ratio

$$\left(\frac{\omega_s}{\omega_p}\right)^n = \frac{\sqrt{10^{-0.1A_s}-1}}{\sqrt{10^{-0.1A_p}-1}} = \frac{\sigma_s}{\sigma_p}$$

$$n\log_{10}\left(\frac{\omega_s}{\omega_p}\right) = \log_{10}\left(\frac{\sigma_s}{\sigma_p}\right)$$

Solving for n

$$n = \frac{\log_{10}\left(\frac{\sigma_s}{\sigma_p}\right)}{\log_{10}\left(\frac{\omega_s}{\omega_p}\right)} \quad if\ \omega_p\ is\ the\ cutoff\ frequency\ n = \frac{\log_{10}\sigma_s}{\log_{10}\left(\frac{\omega_s}{\omega_c}\right)}$$

If $10^{-0.1A_s} \gg 1$ then $\log_{10}\sigma_s \approx -0.05A_s$

$$n = \frac{-0.05A_s}{\log_{10}\left(\frac{\omega_s}{\omega_p}\right)}$$

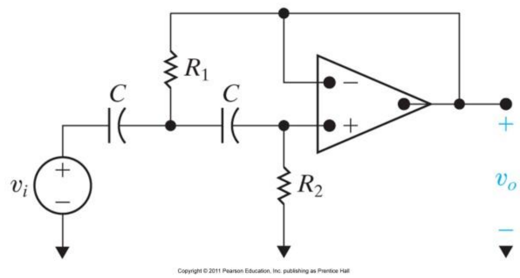
Note: Since the order of the filter must be an integer values when calculating the value always round UP to the nearest integer value.

Butterworth High-pass, Bandpass and Bandreject Filters

Again for the polynomials of a Butterworth filter the transfer function needs to take the form of:

$$H(s) = \frac{s^2}{s^2 + b_1 s + 1} = \frac{V_o}{V_i}$$

$$= \frac{s^2}{s^2 + \frac{2}{R_2 C}s + \frac{1}{R_1 R_2 C^2}}$$



Setting C=1

$$H(s) = \frac{s^2}{s^2 + \frac{2}{R_2}s + \frac{1}{R_1 R_2}}$$

For the variables

$$b_1 = \frac{2}{R_2}\ and\ 1 = \frac{1}{R_1 R_2}$$

Observations

- The high-pass circuit is like the low-pass with the capacitors and resistors switched
- The prototype high-pass filter transfer function can be obtained from the low-pass by replacing s with 1/s.
- By cascading the low- and high-pass Butterworth filter circuits we can obtain the bandpass and bandreject circuits

15.5 Narrowband Bandpass and Bandreject Filters

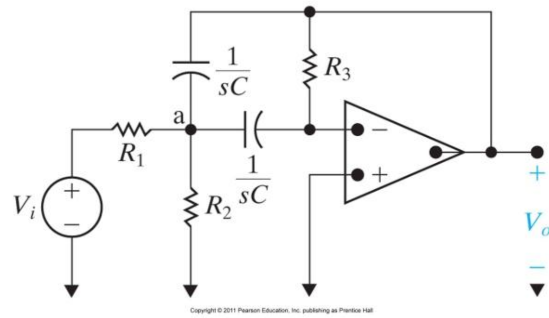
Presently, the methods used to develop Bandpass and Bandreject filters' using cascading low-pass and high-pass filters is only for broadband, or low-Q filters

For $H(s) = \frac{0.5\beta s}{s^2+\beta s+\omega_c^2}$; $\beta = 2\omega_c$; $\omega_o^2 = \omega_c^2$

$$Q = \frac{\omega_o}{\beta} = \frac{1}{2}$$

Thus a quality factor of 0.5 is the largest that can be achieved using this method. (Transfer function has real distinct poles)

A circuit giving complex conjugate poles



Summing the currents at the inverter

$$\frac{V_a}{1/_{sC}} = \frac{-V_o}{R_3}$$

$$V_a = \frac{-V_o}{sR_3C}$$

At node a

$$\frac{V_a - V_i}{R_1} + (V_a - V_o)sC + \frac{V_a}{R_2} + V_a sC = 0$$

Solving for V_i

$$\frac{V_i}{R_1} = \left(\frac{2R_1R_2sC + R_1 + R_2}{R_1R_2}\right)V_a - V_o sC$$

$$V_i = (2R_1sC + R_1 + 1)V_a - V_o sCR_1$$

Substituting V_a

$$V_i = (2R_1sC + R_1 + 1)\left(\frac{-V_o}{sR_3C}\right) - V_o sCR_1$$

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{-s}{R_1C}}{s^2 + \frac{2}{R_3C} + \frac{1}{R_{eq}R_3C^2}} \quad where\ R_{eq} = \frac{R_1R_2}{R_1 + R_2}$$

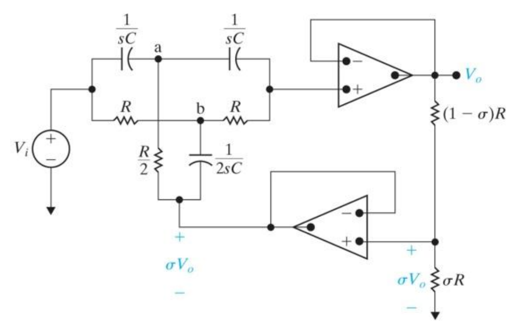
The other parameters are

$$\beta = \frac{2}{R_3C}; \quad K\beta = \frac{1}{R_1C}; \quad \omega_o^2 = \frac{1}{R_{eq}R_3C^2}$$

The prototype version for $\omega_0 = 1$ and C = 1

$$R_1 = \frac{Q}{K}; \quad R_2 = \frac{Q}{2Q^2 - K}; \quad R_3 = 2Q$$

To correct the Low-Q restriction for a Bandreject filter, the *twin-T notch filter* (from dual T-shape at nodes) is used.



Summing current from a

$$(V_a - V_i)sC + (V_a - V_o)sC + \frac{2(V_a - \sigma V_o)}{R} = 0$$

Rewriting

$$V_a(2sCR + 2) - V_o(sCR + 2\sigma) = sCRV_i$$

Summing current from b

$$\frac{V_b - V_i}{R} + (V_b - \sigma V_o)2sC + \frac{V_b - V_o}{R} = 0$$

Rewriting

$$V_b(2sCR + 2) - V_o(1 + 2\sigma RCs) = V_i$$

Summing current from the non-inverted input; top op amp

$$\frac{V_o - V_b}{R} + (V_o - V_a)sC = 0 = -sRCV_a - V_b + (sRC + 1)V_o$$

Solving using Cramer's rule to solve for V_o

$$V_o = \frac{\begin{vmatrix} 2(sCR+1) & 0 & sCRV_i \\ 0 & 2(sCR+1) & V_i \\ -RCs & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2(sCR+1) & 0 & -(sCR+2\sigma) \\ 0 & 2(sCR+1) & -(1+2\sigma RCs) \\ -RCs & -1 & sRC+1 \end{vmatrix}} = \frac{(R^2C^2s^2+1)V_i}{R^2C^2s^2 + 4RC(1-\sigma)s + 1}$$

$$H(s) = \frac{V_o}{V_i} = \frac{s^2 + \frac{1}{R^2C^2}}{s^2 + \frac{4(1-\sigma)}{RC}s + \frac{1}{R^2C^2}} = \frac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$$

Where

$$\omega_0^2 = \frac{1}{R^2C^2}; \quad \beta = \frac{4(1-\sigma)}{RC}$$

Again as in all the designs it will be necessary to choose one of the unknown components. Usually picking a standard capacitor value is best since there is a limited selection available.